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A priori error estimates of finite element solutions of parametrized strongly nonlinear boundary value problems

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Abstract

Nonlinear boundary value problems with parameters are called parametrized nonlinear boundary problems. This paper studies a priori error estimates of finite element solutions of second-order parametrized strongly nonlinear boundary value problems in divergence form on one-dimensional bounded intervals. The Banach space $W_0^{1,\infty}$ is chosen in formulation of the error analysis so that the nonlinear differential operators defined by the differential equations are nonlinear Fredholm operators of index 1. Finite element solutions are defined in a natural way, and several a priori estimates are proved on regular branches and on branches around turning points. In the proofs the extended implicit function theorem due to Brezzi et al. (1980) plays an essential role.

Keywords: Parametrized nonlinear boundary value problems; Fredholm operators; Regular branches; Turning points; Finite element solutions; A priori error estimates

AMS classification: 65L10; 65L60

1. Introduction

Let X, Y be Banach spaces and $\Lambda \subset \mathbb{R}^n$ a bounded interval. Let $F : \Lambda \times X \to Y$ be a smooth operator. The nonlinear equation

$$F(\lambda, u) = 0, \tag{1.1}$$

with parameters $\lambda \in \Lambda$ is called *parametrized nonlinear equations.*

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Let $(\lambda, u) \in \Lambda \times X$ be a solution of (1.1). Intuitively, the set of the solutions of (1.1) would form n-dimensional hypercurves in the Banach space $\mathbb{R}^n \times X$. If $D_uF(\lambda, u) \in \mathscr{L}(X, Y)$, the Fréchet derivative of F with respect to u, is an isomorphism, then, by the implicit function theorem, the above intuition is correct, i.e. there exists a locally unique branch of solutions around (λ, u), and the branch is parametrized by λ. Such branches on which $D_uF(\lambda, u)$ is isomorphism at each (λ, u) are called *regular branches.*

However, if $D_uF(\lambda, u)$ is *not* an isomorphism, the state of equilibrium defined by (1.1) becomes unstable and the behavior of the solutions is unpredictable; the hypercurve of the solutions might be a fold, or there might be several hypercurves of solutions intersecting at that point. The folding points are called *turning points.* The points at which the hypercurves of solutions are intersecting are called *bifurcation points.* (Note that the definition of bifurcation points given by some authors includes turning points.)

In this paper we deal with the parametrized nonlinear equation $F : \Lambda \times H_0^1(J) \to H^{-1}(J)$ with one parameter $\lambda \in \Lambda$ defined as a *nonlinear boundary value problem*

$$F(\lambda, u) = 0, \quad (\lambda, u) \in \Lambda \times H_0^1(J), \tag{1.2}$$

$$\langle F(\lambda, u), v\rangle := \int_J [a(\lambda, x, u'(x))v' + f(\lambda, x, u(x))v]\,\mathrm{d}x, \quad \forall v \in H_0^1(J), \tag{1.3}$$

where $J := (b, c) \subset \mathbb{R}$ is a bounded interval, and $a, f : \Lambda \times J \times \mathbb{R} \to \mathbb{R}$ are sufficiently smooth functions. Since F is a second-order differential operator in divergence form, finite element solutions of (1.2) are defined in a natural way.

Brezzi et al. [3–5] presented a comprehensive work on the numerical analysis of parametrized nonlinear problems. They first proved an extended implicit function theorem with error estimates on Banach spaces. Then, using the implicit function theorem, they obtained several results of a priori error estimates of finite element solutions [3, 4]. In [5], they considered approximation of solution branches around bifurcation points, which will not be dealt with in this paper.

Following [3–5] Fink and Rheinboldt released several papers about numerical analysis of parametrized nonlinear equations ([8, 9, 11], and references therein). While the formulation of [3–5] was rather restrictive, Fink and Rheinboldt developed their theory of a priori error estimates of numerical solutions in a very general setting using the theory of differential geometry.

Fink and Rheinboldt employed the theory of *Fredholm operators.* Let X and Y be Banach spaces and $F : X \to Y$ a differentiable mapping. Then, F is called *Fredholm* on an open set $U \subset X$ if the Fréchet derivative $DF(x)$ satisfies the following conditions at any $x \in U$:

(1) $\dim \operatorname{Ker} DF$ is finite,

(2) $\operatorname{Im} DF$ is closed,

(3) $\dim \operatorname{Coker} DF$ is finite.

We must note that, in the above prior works by Brezzi et al. and Fink–Rheinboldt, only *mildly nonlinear* problems were considered. If $a(\lambda, x, y)$ in (1.3) is nonlinear with respect to y, the operator F is called *strongly nonlinear* (*quasilinear*), otherwise it is called *mildly nonlinear* (*semilinear*).

Following the above prior works, we here develop a thorough theory of a priori and a posteriori error estimates of finite element solutions of (1.2) on regular branches and on branches around turning points in the case that the number of parameters is one, that is, $\Lambda \subset \mathbb{R}$. Since our formulation of parametrized nonlinear equations includes strongly nonliear problems, our theory is an essential extension of the prior works.

In this paper we present the theory of a priori error estimates. In [13] the theory of a posteriori error estimates and several numerical examples will be given. In the following, the outline of this paper is described.

First, we show that the exact and finite element solutions of (1.2) form one-dimensional smooth manifolds. If F is mildly nonlinear, showing that solutions form manifolds would not be very difficult. If F is strongly nonlinear, however, it would become very difficult, or F would not be even differentiable in $\Lambda \times H_0^1(J)$.

Therefore, we redefine (1.2) and (1.3) using the Sobolev space $W^{1,\infty}(J)$. Then, F becomes as smooth as the functions a and f, and it is a Fredholm operator in a certain open set. From the Fink–Rheinboldt theory, we conclude that the exact and finite element solutions form smooth manifolds under suitable conditions.

Next, we prove several a priori estimates of finite element solution manifolds of (1.2) using the extended implicit function theorem due to Brezzi et al. [3]. As mentioned before, we need to take the Sobolev space $W^{1,\infty}(J)$ as the stage of the error analysis of finite element solution manifolds. However, using $W^{1,\infty}(J)$ in the formulation make the finite element analysis difficult. So we have to come up with several new tricks to overcome this difficulty. The following is the most essential trick.

Since our operator F is defined on $W^{1,\infty}(J)$, its Fréchet derivative D_uF is a linear operator on $W^{1,\infty}(J)$. However, D_uF can be extended to an element of $\mathscr{L}(H_0^1, H^{-1})$ and thus the usual theory of finite element can be applied to D_uF.

Another new idea is 'rotation' or 'pivoting' of the coordinate to handle turning points. In [4], a slightly different formulation from that of [3] was used to deal with turning points. In Fink–Rheinboldt's theory, certain isomorphisms were introduced in the formulation so that both regular branches and branches around turning points were treated simultaneously. In this paper, we put an auxiliary equation in the original problem (1.2) or (1.3) so that the enlarged operator is an isomorphism between Banach spaces around turning points or on 'steep slope'. Then we do the same thing what we do on regular branches to the extended operator.

In this paper one-dimensional case is discussed. Under certain assumptions the results obtained here will be extended to two-dimensional case in [14].

This paper is a revision of a part of one of the author's Ph.D. dissertation [12].

2. Preliminary

In this section we prepare notation and a necessary lemma.

Let $J := (b, c) \subset \mathbb{R}$ be a bounded interval. For a positive integer m and a real $p \in [1, \infty]$, we denote by $W^{m,p}(J)$ the usual L^p-Sobolev space of order m, that is,

$$W^{m,p}(J) := \{u \in L^p(J) \mid D^k u \in L^p(J),\ 0 \leqslant k \leqslant m\}.$$

We define the norm of $W^{m,p}(J)$ by

$$\|u\|_{W^{m,p}} := \sum_{k=0}^{m} \|D^k u\|_{L^p}.$$

For $p \in [1, \infty]$, we define the closed subspace $W_0^{1,p}(J)$ by

$$W_0^{1,p}(J) := \{u \in W^{1,p}(J) \mid u = 0 \text{ on } \partial J\}.$$

As usual, we denote $W^{m,2}(J)$ and $W_0^{1,2}(J)$ by $H^m(J)$ and $H_0^1(J)$, respectively.

Note that $C_0^\infty(J)$, the set of infinitely many times differentiable functions with compact supports, is dense in $W_0^{1,p}(J)$ for p, $1 \leqslant p < \infty$, but if $p = \infty$, $C_0^\infty(J)$ is *not* dense in $W_0^{1,\infty}(J)$.

By the Poincaré inequality, the norm

$$\|u\|_{W_0^{1,p}} := \|u'\|_{L^p} \tag{2.1}$$

is equivalent to the norm $\|\cdot\|_{W^{1,p}}$ in $W_0^{1,p}(J)$. We always take the norm (2.1) for $W_0^{1,p}(J)$ in this paper.

For $1 \leqslant q < \infty$ and p with $1/p + 1/q = 1$, let $W^{-1,p}(J)$ be the dual space of $W_0^{1,q}(J)$ with the norm

$$\|F\|_{W^{-1,p}} := \sup_{\|x\|_{W_0^{1,q}} \leqslant 1} |{}_p\langle F, x\rangle_q|, \quad F \in W^{-1,p}(J),$$

where ${}_p\langle \cdot, \cdot \rangle_q$ is the duality paring between $W^{-1,p}(J)$ and $W_0^{1,q}(J)$. Then we have

Lemma 2.1. *For any $F \in W^{-1,p}(J)$ with $1 < p \leqslant \infty$, there exists a unique $u \in W_0^{1,p}(J)$ so that*

$${}_p\langle F, v\rangle_q = \int_J u'v' \, dx, \quad \forall v \in W_0^{1,q}(J).$$

Lemma 2.1 is a direct consequence of [2, Proposition VIII.13].

In this paper, we omit (J) from the notation of Sobolev spaces when there is no danger of confusion. Also, we write $\langle \cdot, \cdot \rangle$ instead of ${}_p\langle \cdot, \cdot \rangle_q$ when the setting of the duality paring is obvious.

Subscripts like a_y and f_λ stand for partial derivatives with respect to y and λ, respectively.

3. Formulation of the problem

In this section we formulate our problem rigorously. To do this we define the nonlinear operator $F : \Lambda \times W_0^{1,\infty} \to W^{-1,\infty}$ by, for $\lambda \in \Lambda \subset \mathbb{R}$ and $u \in W_0^{1,\infty}$,

$$\langle F(\lambda, u), v\rangle := \int_J [a(\lambda, x, u'(x))v'(x) + f(\lambda, x, u(x))v(x)] \, dx, \quad \forall v \in W_0^{1,1}, \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $W^{-1,\infty}$ and $W_0^{1,1}$.

For F being well-defined and smooth we require several conditions for a and f.

A function $\psi : \Lambda \times J \times \mathbb{R} \to \mathbb{R}$ is called *Carathéodory continuous* if ψ satisfies the following conditions: for $(\lambda, x, y) \in \Lambda \times J \times \mathbb{R}$,

$\psi(\lambda, x, y)$ is continuous with respect to λ and y for almost all x,

$\psi(\lambda, x, y)$ is Lebesgue measurable with respect to x for all λ and y.

If $\psi(\lambda,x,y)$ is Carathéodory continuous, $\psi(\lambda,x,u(x))$ is Lebesgue measurable with respect to x for any Lebesgue measurable function u (see [1, p. 76]).

Let $\alpha=(\alpha_1,\alpha_2)$ be usual multiple index with respect to λ and y. That is, for $\alpha=(\alpha_1,\alpha_2)$, $D^\alpha a(\lambda,x,y)$ means $(\partial^{|\alpha|}/\partial\lambda^{\alpha_1}\partial y^{\alpha_2})a(\lambda,x,y)$.

Let $d\geqslant 1$ be an integer. For α, $|\alpha|\leqslant d$, we define the maps $\mathbb{A}^\alpha(\lambda,u)$ and $\mathbb{F}^\alpha(\lambda,u)$ for $(\lambda,u)\in\Lambda\times W_0^{1,\infty}$ by

$$\mathbb{A}^\alpha(\lambda,u)(x):=D^\alpha a(\lambda,x,u'(x)), \tag{3.2}$$

$$\mathbb{F}^\alpha(\lambda,u)(x):=D^\alpha f(\lambda,x,u(x)). \tag{3.3}$$

We then assume that

Assumption 3.1. *For all α, $|\alpha|\leqslant d$, we suppose that*

(1) *For almost all $x\in J$, $D^\alpha a(\lambda,x,y)$ and $D^\alpha f(\lambda,x,y)$ exist at all $(\lambda,y)\in\Lambda\times\mathbb{R}$, and they are Carathéodory continuous.*

(2) *The mapping $\mathbb{A}^\alpha$ defined by* (3.2) *is a continuous operator from $\Lambda\times W_0^{1,\infty}$ to L^∞, and the image $\mathbb{A}^\alpha(U)\subset L^\infty$ of any bounded subset $U\subset\Lambda\times W_0^{1,\infty}$ is bounded.*

(3) *The mapping $\mathbb{F}^\alpha$ defined by* (3.3) *is a continuous operator from $\Lambda\times W_0^{1,\infty}$ to L^1, and the image $\mathbb{F}^\alpha(U)\subset L^1$ of any bounded subset $U\subset\Lambda\times W_0^{1,\infty}$ is bounded.*

Assumption 3.1 is satisfied if $a,f:\Lambda\times J\times\mathbb{R}\to\mathbb{R}$ are, for instance, C^d functions. By simple computation we obtain the following lemma.

Lemma 3.2. *Suppose that a and f satisfy Assumption* 3.1. *Then F defined by* (3.1) *is a C^d mapping. Its Fréchet derivatives are written as*

$$\langle D_uF(\lambda,u)\psi,v\rangle=\int_J[a_y(\lambda,x,u'(x))\psi'v'+f_y(\lambda,x,u(x))\psi v]\,\mathrm{d}x,$$

$$\langle D_\lambda F(\lambda,u)\eta,v\rangle=\eta\int_J[a_\lambda(\lambda,x,u'(x))v'+f_\lambda(\lambda,x,u(x))v]\,\mathrm{d}x,$$

for $\psi\in W_0^{1,\infty}$, $v\in W_0^{1,1}$, and $\eta\in\mathbb{R}$. Moreover, we have the following estimates:

$$\|D_uF(\lambda,u)\|_{\mathscr{L}(W_0^{1,\infty},W^{-1,\infty})}\leqslant\|a_y(\lambda,x,u'(x))\|_{L^\infty}+\|f_y(\lambda,x,u(x))\|_{L^1},$$

$$\|D_\lambda F(\lambda,u)\|_{W^{-1,\infty}}\leqslant\|a_\lambda(\lambda,x,u'(x))\|_{L^\infty}+\|f_\lambda(\lambda,x,u(x))\|_{L^1}.$$

Now, we define our problem.

Problem 3.3. *Under Assumption* 3.1 *with $d\geqslant 1$, solve the following equation*: *Find $\lambda\in\Lambda$ and $u\in W_0^{1,\infty}$ such that*

$$\langle F(\lambda,u),v\rangle=0,\quad\forall v\in W_0^{1,1},$$

where F is defined by (3.1).

4. Fredholm operator and the solution manifold

In this section we prove that, if $a(\lambda,x,y)$ satisfies certain conditions, F will be a nonlinear Fredholm operator and solutions of Problem 3.3 form a one-dimensional differentiable manifold. The following lemmas are essential.

Let $p\in(1,\infty]$ and $\alpha\in L^\infty$. Define $A : W_0^{1,p} \to W^{-1,p}$ by

$$\langle Au,v\rangle := \int_J \alpha(x)u'(x)v'(x)\,\mathrm{d}x, \quad \forall v\in W_0^{1,q}, \tag{4.1}$$

where $1/p+1/q=1$, and $\langle\cdot,\cdot\rangle$ is the duality pairing between $W^{-1,p}$ and $W_0^{1,q}$. Then we have

Lemma 4.1. *Suppose $\alpha^{-1}\in L^\infty$ and $\int_J \mathrm{d}x/\alpha(x)\neq 0$. Then A is an isomorphism between $W_0^{1,p}$ and $W^{-1,p}$.*

Proof. First, we prove that A is onto. Let $c_0 := \int_J \mathrm{d}x/\alpha(x)$. Take an arbitrary $F\in W^{-1,p}$. By Lemma 2.1 we know that there exists a unique $\psi\in W_0^{1,p}$ such that

$$\langle F,v\rangle = \int_J \psi'(x)v'(x)\,\mathrm{d}x, \quad \forall v\in W_0^{1,q}.$$

Let

$$c_1 := -\frac{1}{c_0}\int_J \frac{\psi'(x)}{\alpha(x)}\,\mathrm{d}x \quad \text{and} \quad u(x) := \int_b^x \frac{\psi'(t)+c_1}{\alpha(t)}\,\mathrm{d}t.$$

Then it follows that $u\in W_0^{1,p}$, and $\langle Au,v\rangle = \langle F,v\rangle$ for all $v\in W_0^{1,q}$. Hence, A is onto.

Next, we show that A is one-to-one. Suppose that $u_1,u_2\in W_0^{1,p}$ and

$$\int_J \alpha(x)u_1'(x)v'(x)\,\mathrm{d}x = \int_J \alpha(x)u_2'(x)v'(x)\,\mathrm{d}x, \quad \forall v\in W_0^{1,q}.$$

By [2, Lemma VIII.1.], there is a constant c_2 such that $\alpha(x)(u_1'(x)-u_2'(x))=c_2$ for almost all $x\in J$. Since

$$0=\int_J (u'(x)-u_2'(x))\,\mathrm{d}x = c_2\int_J \frac{\mathrm{d}x}{\alpha(x)} = c_0c_2,$$

and $c_0\neq 0$, we conclude that $u_1'(x)=u_2'(x)$ and $u_1=u_2$. That is, A is one-to-one.

Since A is continuous and bijective, A^{-1} is also bounded by the closed graph theorem. Therefore, A is an isomorphism between $W_0^{1,p}$ and $W^{-1,p}$. □

Lemma 4.2. *Suppose that $\alpha^{-1}\in L^\infty$ and $\int_J \mathrm{d}x/\alpha(x)=0$. Then*

(1) $\dim\operatorname{Ker}A = 1$ *and* $\operatorname{Ker}A = \{\varphi\in W_0^{1,p} \mid \varphi'(x)=c_0\alpha(x)^{-1},\ c_0\in\mathbb{R}\}$,

(2) $\operatorname{Im}A\subset W^{-1,p}$ *is closed,*

(3) $\dim\operatorname{Coker}A = 1$.

Proof. (1) Let $\varphi(x) := \int_b^x \mathrm{d}t/\alpha(t)$. Then, by the assumption, we have $\varphi\in W_0^{1,\infty}\subset W_0^{1,p}$, and $\langle A\varphi,v\rangle = 0$ for all $v\in W_0^{1,q}$. Therefore, $\varphi\in\operatorname{Ker}A$.

Conversely, for any $u \in \operatorname{Ker} A$, there is a constant c_0 such that $\alpha(x)u'(x) = c_0$ for almost all $x \in J$. This implies that $u = c_0\varphi$. Hence, (1) is proved.

(2) First, we define the subset $X \subset W_0^{1,p}$ by

$$X := \left\{ \psi \in W_0^{1,p} \;\middle|\; \int_J \frac{\psi'(x)}{\alpha(x)}\,\mathrm{d}x = 0 \right\}$$

Clearly, X is a closed subspace of $W_0^{1,p}$.

Let $T \in \mathscr{L}(W_0^{1,p}, W^{-1,p})$ be the isomorphism defined by $\langle Tu, v\rangle := \int_J u'v'\,\mathrm{d}x$, $\forall v \in W_0^{1,q}$. Let $\tilde{X} := T(X)$. Take any $\psi \in X$, and define $u(x) := \int_b^x [\psi'(t)/\alpha(t)]\,\mathrm{d}t$. Then, we have $u \in W_0^{1,p}$, and $\langle Au, v\rangle = \langle T\psi, v\rangle$, for all $v \in W_0^{1,q}$. Hence, we have that $\operatorname{Im} A \supset \tilde{X}$.

Now, take any $\eta \in W_0^{1,p}$ and define γ by $\gamma(x) := \int_b^x (\alpha(t)\eta'(t) - c_1)\,\mathrm{d}t$, where $c_1 := \int_J \alpha\eta'\mathrm{d}x/|J|$. We check that $\gamma \in W_0^{1,p}$ and $T\gamma = A\eta$. Moreover, we have $\gamma \in X$ because

$$\int_J \frac{\gamma'(x)}{\alpha(x)}\,\mathrm{d}x = \int_J \eta'(x)\,\mathrm{d}x - c_1 \int_J \frac{\mathrm{d}x}{\alpha(x)} = 0.$$

Hence, we conclude that $\operatorname{Im} A = \tilde{X}$ and $\operatorname{Im} A$ is closed.

(3) As before, define $\psi_0 \in W_0^{1,p}$ by $\psi_0(x) := \int_b^x \alpha(t)^{-1}\,\mathrm{d}t$. Since $\int_J (\psi_0'/\alpha)\,\mathrm{d}x = \int_J \alpha^{-2}\,\mathrm{d}x \neq 0$, we have $\psi_0 \notin X$. Let $c_2 := \int_J \alpha^{-2}\,\mathrm{d}x > 0$. Take any $\psi \in W_0^{1,p}$ and let $c_3 := \int_J (\psi'/\alpha)\,\mathrm{d}x$. Then $\psi - (c_3/c_2)\psi_0 \in X$ because

$$\int_J \frac{\psi'(x) - (c_3/c_2)\psi_0'(x)}{\alpha(x)}\,\mathrm{d}x = \int_J \frac{\psi'(x)}{\alpha(x)}\,\mathrm{d}x - \frac{c_3}{c_2}\int_J \frac{\mathrm{d}x}{\alpha(x)^2} = 0.$$

This implies that for any $\psi \in W_0^{1,p}$ there exist $c_4 \in \mathbb{R}$ and $\psi_1 \in X$ such that $\psi = c_4\psi_0 + \psi_1$. The uniqueness of such decomposition is obvious.

Therefore, we showed that $W^{-1,p} = \operatorname{Im} A \oplus \operatorname{span}\{T\psi_0\}$, and (3) is proved. □

From Lemmas 4.1 and 4.2 and the definition of Fredholm operators, we finally obtain

Theorem 4.3. *If $\alpha^{-1} \in L^\infty$, then the linear operator A defined by* (4.1) *is a Fredholm operator and* $\operatorname{ind} A$, *the index of A, is* 0.

Let us now return to our main problem. We define the subset $\mathscr{S} \subset \Lambda \times W_0^{1,\infty}$ by

$$\mathscr{S} := \{(\lambda, u) \in \Lambda \times W_0^{1,\infty} \mid a_y(\lambda, x, u'(x))^{-1} \in L^\infty\}. \tag{4.2}$$

Since the mapping $\Lambda \times W_0^{1,\infty} \ni (\lambda, u) \mapsto a_y(\lambda, x, u'(x)) \in L^\infty$ is continuous, we have

Lemma 4.4. *If a and f satisfy Assumption* 3.1 *with $d \geqslant 1$, $\mathscr{S}$ is an open set in $\Lambda \times W_0^{1,\infty}$.*

Now, from the standard theory of Fredholm operators, we obtain the following theorem.

Theorem 4.5. *Suppose that a and f satisfy Assumption* 3.1 *with $d \geqslant 1$. Then in $\mathscr{S}$, the operator $F : \mathscr{S} \to W^{-1,\infty}$ defined by* (3.1) *is a nonlinear Fredholm operator of index* 1.

Proof. From Lemma 3.2 and Theorem 4.3, the operator $D_uF(\lambda,u): W_0^{1,\infty} \to W^{-1,\infty}$ is Fredholm and its index is 0 for $(\lambda,u)\in\mathscr{S}$. Since $DF(\lambda,u):\mathbb{R}\times W_0^{1,\infty} \to W^{-1,\infty}$ is written as $DF(\lambda,u)(\eta,\psi)= D_uF(\lambda,u)\psi+\eta D_\lambda F(\lambda,u)$ for $\eta\in\mathbb{R}$ and $\psi\in W_0^{1,\infty}$, Theorem 4.5 is concluded. □

We define the subset $\mathscr{R}(F,\mathscr{S})\subset\mathscr{S}$ by

$$\mathscr{R}(F,\mathscr{S}):=\{(\lambda,u)\in\mathscr{S}\,|\,DF(\lambda,u) \text{ is onto}\}. \tag{4.3}$$

and have

Lemma 4.6. *For any* $(\lambda,u)\in\mathscr{R}(F,\mathscr{S})$, $\dim\operatorname{Ker}D_uF(\lambda,u)$ *is at most* 1.

Proof. Assume that $N:=\dim\operatorname{Ker}D_uF(\lambda,u)\geqslant 2$ for some $(\lambda,u)\in\mathscr{R}(F,\mathscr{S})$. Note that the elements of $\operatorname{Ker}DF(\lambda,u)$ are solutions of the linearized equation

$$DF(\lambda,u)(\mu,\psi)=\mu D_\lambda F(\lambda,u)+D_uF(\lambda,u)\psi=0,\quad \mu\in\mathbb{R},\ \psi\in W_0^{1,\infty}. \tag{4.4}$$

If $D_\lambda F(\lambda,u)\notin\operatorname{Im}D_uF(\lambda,u)$, by (4.4), we obtain

$$\operatorname{Ker}DF(\lambda,u)=\{(0,\phi)\in\Lambda\times W_0^{1,\infty}\,|\,\phi\in\operatorname{Ker}D_uF(\lambda,u)\}$$

and $\dim\operatorname{Ker}DF(\lambda,u)=N>1$. This contradicts to (4.3) and $\operatorname{ind}F=1$.

Therefore, we should conclude that $D_\lambda F(\lambda,u)\in\operatorname{Im}D_uF(\lambda,u)$. Let $\psi_0\in W_0^{1,\infty}$ be such that $D_\lambda F(\lambda,u)=D_uF(\lambda,u)\psi_0$. Then we obtain

$$\operatorname{Ker}DF(\lambda,u)=\{(\mu,-\mu\psi_0+\phi)\in\Lambda\times W_0^{1,\infty}\,|\,\mu\in\mathbb{R},\phi\in\operatorname{Ker}D_uF(\lambda,u)\}, \tag{4.5}$$

and hence $\dim\operatorname{Ker}DF(\lambda,u)=N+1>1$. Therefore, we get a contradiction again, and Lemma 4.6 is proved. □

The elements of $\mathscr{R}(F,\mathscr{S})$ are called *regular points*. The elements of $F(\mathscr{R}(F,\mathscr{S}))$ are called *regular values*.

By Theorem 4.5, we can apply the Fink–Rheinboldt theory [8, 9, 11] to the operator F and obtain the main theorem of this section.

Theorem 4.7. *Suppose that a and f satisfy Assumption* 3.1 *with $d\geqslant 1$. Let $e\in F(\mathscr{R}(F,\mathscr{S}))$. Then*

$$\mathscr{M}=\mathscr{M}_e:=\{(\lambda,u)\in\mathscr{R}(F,\mathscr{S})\,|\,F(\lambda,u)=e\}$$

is a one-dimensional C^d-manifold without boundary. Moreover, for each $(\lambda,u)\in\mathscr{M}$, the tangent space $T_{(\lambda,u)}\mathscr{M}$ at (λ,u) is $\operatorname{Ker}DF(\lambda,u)$.

Therefore, if $0\in F(\mathscr{R}(F,\mathscr{S}))$, the solutions of Problem 3.3 *form a one-dimensional C^d-manifold without boundary in $\mathscr{R}(F,\mathscr{S})$.*

In the sequel of this paper we always assume that $0\in F(\mathscr{R}(F,\mathscr{S}))$.

Now, let us consider the linearized equation (4.4). From Lemma 4.6, we would have four cases for $(\lambda,u)\in\mathscr{R}(F,\mathscr{S})$.

Case 1. $\operatorname{Ker}D_uF(\lambda,u)=\{0\}$ and $D_\lambda F(\lambda,u)\in\operatorname{Im}D_uF(\lambda,u)$.

In this case, by the implicit function theorem, there exists a unique C^d map $\Lambda \ni \lambda \mapsto u(\lambda) \in W_0^{1,\infty}$ such that $F(\lambda, u(\lambda)) = 0$ for any λ. Hence, this case corresponds to *regular branches*.

Case 2. $\dim \operatorname{Ker} D_u F(\lambda, u) = 1$ and $D_\lambda F(\lambda, u) \notin \operatorname{Im} D_u F(\lambda, u)$.

In Case 2, using the well-known *Liapunov–Schmidt reduction* (see, for instance, [10]), we can show that this case corresponds to (general) *turning points*.

Case 3. $\operatorname{Ker} D_u F(\lambda, u) = \{0\}$ and $D_\lambda F(\lambda, u) \notin \operatorname{Im} D_u F(\lambda, u)$.

By a similar argument to the proof of Lemma 4.6, we see that this case cannot happen.

Case 4. $\dim \operatorname{Ker} D_u F(\lambda, u) = 1$ and $D_\lambda F(\lambda, u) \in \operatorname{Im} D_u F(\lambda, u)$.

By (4.5), we have $\dim \operatorname{Ker} DF(\lambda, u) = 2$ and $\dim \operatorname{Coker} DF(\lambda, u) = 1$. Hence, this is not the case for $(\lambda, u) \in \mathscr{R}(F, \mathscr{S})$. In this case we may have a bifurcation phenomenon.

By the above consideration we now know that

$$(\lambda, u) \in \mathscr{R}(F, \mathscr{S}) \overset{\text{def}}{\Longleftrightarrow} (\lambda, u) \in \mathscr{S} \text{ and } DF(\lambda, u) \in \mathscr{L}(\mathbb{R} \times W_0^{1,\infty}, W^{-1,\infty}) \text{ is onto,}$$
$$\Rightarrow \quad \text{we have either Case 1 or Case 2.}$$

5. Regularity of solutions

In this section we examine the regularity of the solutions $(\lambda, u) \in \mathscr{M}_0$. To do this we need additional assumptions. Let p^*, $2 \leqslant p^* \leqslant \infty$ be taken and fixed.

Assumption 5.1. *Under Assumption* 3.1 *with* $d \geqslant 1$, *we assume that*

(1) *For all* $\lambda \in \Lambda$, *the functions* $a(\lambda, \cdot, \cdot)$, $a_y(\lambda, \cdot, \cdot) : J \times \mathbb{R} \to \mathbb{R}$ *are continuous.*

(2) *For all* $(\lambda, y) \in \Lambda \times \mathbb{R}$, *there exist* $a_x(\lambda, x, y)$ *for almost all* $x \in J$ *and are Carathéodory continuous.*

(3) *The composition functions* $f(\lambda, x, u(x))$, $a_x(\lambda, x, u'(x))$ *are in* L^{p^*} *for any* $(\lambda, u) \in \Lambda \times W_0^{1,\infty}$. *Moreover, for any bounded subsets* $K \subset \Lambda \times W_0^{1,\infty}$,

$$\{f(\lambda, x, u(x)) \in L^{p^*} \mid (\lambda, u) \in K\}, \quad \{a_x(\lambda, x, u'(x)) \in L^{p^*} \mid (\lambda, u) \in K\}$$

are bounded in L^{p^*}.

Lemma 5.2. *Let* $(\lambda, u) \in \mathscr{M}_0$. *Suppose that Assumptions* 3.1 *and* 5.1 *hold. Then* $u \in C^1(\bar{J})$.

Proof. Define f_0 by $f_0(x) := -f(\lambda, x, u(x))$. By Assumption 5.1(3), we have $f_0 \in L^{p^*}$. Now, consider the following equation:

$$\int_J \Phi'(x) v'(x) \, dx = \int_J f_0(x) v(x) \, dx, \quad \forall v \in H_0^1. \tag{5.1}$$

There exists a unique solution $\Phi \in W^{2,p^*}$ of (5.1). Thus, we have

$$a(\lambda, x, u'(x)) = H(x) \in W^{1,p^*}, \tag{5.2}$$

where $H(x) := \Phi'(x) + c_1$ with some constant c_1.

Now, for a fixed $\lambda \in \Lambda$, define the function $G: J \times \mathbb{R} \to \mathbb{R}$ by $G(x,y) := a(\lambda,x,y) - H(x)$. Note that $G_y(x,y) = a_y(\lambda,x,y)$ and, by Assumption 5.1(1), G and G_y are continuous. Also we remark that, for almost all $x_0 \in J$ and $y_0 := u'(x_0)$, we have $G(x_0,y_0) = a(\lambda,x_0,u'(x_0)) - H(x_0) = 0$ and $a_y(\lambda,x_0,y_0) = a_y(\lambda,x_0,u'(x_0)) \neq 0$ because $(\lambda,u) \in \mathcal{M} \subset \mathcal{S}$ and (4.2).

Therefore, by the implicit function theorem [7, Theorem 15.1], we conclude that, in the each neighborhood of x_0, there exists a unique continuous function T such that $T(x_0) = u'(x_0)$ and

$$G(x,T(x)) = a(\lambda,x,T(x)) - H(x) = 0.$$

This means that $T(x) = u'(x)$. Hence, $u'(x)$ is continuous for all $x \in \bar{J}$. □

The following is the main theorem of this section.

Theorem 5.3. *Under Assumptions* 3.1 *and* 5.1, *we have* $u \in W^{2,p^*}$ *for all* $(\lambda,u) \in \mathcal{M}_0$. *Moreover, for all bounded closed subsets* $\widetilde{\mathcal{M}} \subset \mathcal{M}_0$, *there exists a constant* $K(\widetilde{\mathcal{M}})$ *such that* $\sup_{(\lambda,u)\in\widetilde{\mathcal{M}}} \|u\|_{W^{2,p^*}} \leqslant K(\widetilde{\mathcal{M}})$.

Proof. Let $H \in W^{1,p^*}$ be defined by (5.2). For small $\delta > 0$ we write

$$\frac{H(x+\delta)-H(x)}{\delta} = a_y(\lambda,x+\delta,u'(x+\delta)+\varepsilon(u'(x+\delta)-u'(x)))\frac{u'(x+\delta)-u'(x)}{\delta} + \frac{a(\lambda,x+\delta,u'(x))-a(\lambda,x,u'(x))}{\delta}$$

with $0 < \varepsilon < 1$. Since $H \in W^{1,p^*}$, H' exists at almost all $x \in J$ and $H' \in L^{p^*}$. From $(\lambda,u) \in \mathcal{M}_0 \subset \mathcal{S}$, it follows that $a_y(\lambda,x+\delta,u'(x+\delta))^{-1} \in L^\infty$, that is,

$$|a_y(\lambda,x+\delta,u'(x+\delta))| \geqslant \gamma > 0 \quad \text{for any } x+\delta \in J.$$

By Lemma 5.2, we have $u'(x+\delta) \to u'(x)$ as $\delta \to 0$. Hence we obtain

$$|a_y(\lambda,x+\delta,u'(x+\delta)+\varepsilon(u'(x+\delta)-u'(x)))| > 0 \tag{5.3}$$

for all $x \in J$ and sufficiently small $\delta > 0$, and

$$\lim_{\delta\to 0} a_y(\lambda,x+\delta,u'(x+\delta)+\varepsilon(u'(x+\delta)-u'(x))) = a_y(\lambda,x,u'(x)) \tag{5.4}$$

because of Assumption 5.1(1). By (5.3), (5.4) and Assumption 5.1(2), we conclude that, for almost all $x \in J$, $\lim_{\delta\to 0}(u'(x+\delta)-u'(x))/\delta$ exists and

$$u''(x) = \lim_{\delta\to 0} \frac{u'(x+\delta)-u'(x)}{\delta} = \frac{H'(x)-a_x(\lambda,x,u'(x))}{a_y(\lambda,x,u'(x))}. \tag{5.5}$$

Since $H', a_x(\lambda,x,u'(x)) \in L^{p^*}$, we obtain $u'' \in L^{p^*}$ and $u \in W^{2,p^*}$.

Now, let $\widetilde{\mathcal{M}} \subset \mathcal{M}_0$ be a bounded closed subset. Then, we have

$$\sup\{\|a_y(\lambda,x,u'(x))^{-1}\|_{L^\infty}; (\lambda,u) \in \widetilde{\mathcal{M}}\} < \infty. \tag{5.6}$$

The last part of Lemma 5.3 is obtained by (5.5), (5.6), and Assumption 5.1(3). □

6. Finite element solution manifold

Recall that we are considering

Problem 6.1. *Find $\lambda \in \Lambda$ and $u \in W_0^{1,\infty}$ such that*

$$\langle F(\lambda,u),v\rangle = 0, \quad \forall v \in W_0^{1,1}. \tag{6.1}$$

Naturally, we define the finite element solution of (6.1) in the following way. First, we triangulate the interval J into disjoint union of small intervals. Then, we set the finite element space $\mathring{S}_h \subset W_0^{1,\infty} \subset W_0^{1,1}$ using the triangulation. The space of piecewise linear functions on the triangulation is an example of $\mathring{S}_h$. We define the finite element solutions of Problem 6.1 by

Problem 6.1$_{\mathrm{FE}}$. *Find $\lambda_h \in \Lambda$ and $u_h \in \mathring{S}_h$ such that*

$$\langle F(\lambda_h,u_h),\ v_h\rangle = 0, \quad \forall v_h \in \mathring{S}_h.$$

Then, using the Fink and Rheinboldt theory, we will show that the solutions of Problem 6.1$_{\mathrm{FE}}$ also form a differentiable manifold.

Let $(\cdot,\cdot)$ be the inner product of H_0^1 defined by $(u,v) := \int_J u'v'\,\mathrm{d}x$ for $u,v \in H_0^1$. Since $\mathring{S}_h \subset H_0^1$, we define the canonical projection $\Pi_h : H_0^1 \to \mathring{S}_h$ by $(\psi - \Pi_h\psi, v_h) = 0$, $\forall v_h \in \mathring{S}_h$ for $\psi \in H_0^1$.

We see the following equivalences. Define an isomorphism $T \in \mathscr{L}(W^{-1,\infty}, W_0^{1,\infty})$ by $\langle \eta, v\rangle = (T\eta, v)$, $\forall v \in W_0^{1,1}$ for $\eta \in W^{-1,\infty}$. Then, we observe that, for any $v_h \in \mathring{S}_h$ and $v \in H_0^1$,

$$\begin{aligned}\langle F(\lambda_h,u_h),v_h\rangle = 0 &\Leftrightarrow \langle F(\lambda_h,u_h),\Pi_h v\rangle = 0\\ &\Leftrightarrow (TF(\lambda_h,u_h),\Pi_h v) = 0\\ &\Leftrightarrow (\Pi_h TF(\lambda_h,u_h),v) = 0\\ &\Leftrightarrow \langle T^{-1}\Pi_h TF(\lambda_h,u_h),v\rangle = 0.\end{aligned} \tag{6.2}$$

Since H_0^1 is dense in $W_0^{1,1}$, we conclude that Problem 6.1$_{\mathrm{FE}}$ is equivalent to

Problem 6.1$^*_{\mathrm{FE}}$. *Find $\lambda_h \in \Lambda$ and $u_h \in \mathring{S}_h$ such that*

$$\langle F_h(\lambda_h,u_h),v\rangle = 0, \quad \forall v \in W_0^{1,1},$$

where $P_h := T^{-1}\Pi_h T \in \mathscr{L}(W^{-1,\infty}, W^{-1,\infty})$ and $F_h(\lambda_h,u_h) := P_hF(\lambda_h,u_h)$.

Our formulation of Problem 6.1$^*_{\mathrm{FE}}$ seems to depend on T and Π_h. However, we claim that, even if we take other pair (T_α, Π_h^α), and define the finite element solutions by $\langle T_\alpha^{-1}\Pi_h^\alpha T_\alpha F(\lambda_h,u_h),v\rangle = 0$ for all $v \in W_0^{1,1}$, this formulation is equivalent to Problem 6.1$^*_{\mathrm{FE}}$.

Let $\alpha \in L^\infty$ be such that $\alpha(x) \geqslant \varepsilon > 0$ for all $x \in J$, where ε is a constant. Let $(\cdot,\cdot)_\alpha$ be the inner product of H_0^1 defined by $(u,v)_\alpha := \int_J \alpha u'v'\,\mathrm{d}x$ for $u,v \in H_0^1$. Define the isomorphism $T_\alpha \in \mathscr{L}(W^{-1,\infty}, W_0^{1,\infty})$ by $\langle \eta, v\rangle = (T_\alpha\eta, v)_\alpha$, $\forall v \in W_0^{1,1}$ for $\eta \in W^{-1,\infty}$. Also, define the canonical projection $\Pi_h^\alpha : H_0^1 \to \mathring{S}_h$ by $(\psi - \Pi_h^\alpha\psi, v_h)_\alpha = 0$, $\forall v_h \in \mathring{S}_h$ for $\psi \in H_0^1$. By the same manner as in (6.2), we observe that, for any $v_h \in \mathring{S}_h$ and any $v \in H_0^1$,

$$\langle F(\lambda_h,u_h),v_h\rangle = 0 \Leftrightarrow \langle T_\alpha^{-1}\Pi_h^\alpha T_\alpha F(\lambda_h,v_h),v\rangle = 0.$$

Therefore, with the definition $P_h^\alpha := T_\alpha^{-1}\Pi_h^\alpha T_\alpha$, we conclude that

$$P_h F(\lambda_h, u_h) = 0 \Leftrightarrow P_h^\alpha F(\lambda_h, u_h) = 0. \tag{6.3}$$

Hence, our claim is demonstrated.

We will see that these observation is very important for our a priori error estimates because (6.3) guarantees that we can take any $\alpha \in L^\infty$ (that is, (T_α, Π_h^α)) such that $\alpha \geqslant \varepsilon > 0$ in our error analysis.

In the statement of Problem 6.1^*_{FE}, we defined $F_h : \Lambda \times \mathring{S}_h \to W^{-1,\infty}$. Following the Fink–Rheinboldt theory we extend F_h to $\Lambda \times W_0^{1,\infty}$. Define $\overline{F}_h^\alpha : \Lambda \times W_0^{1,\infty} \to W^{-1,\infty}$ by

$$\overline{F}_h^\alpha(\lambda, u) := (I - P_h^\alpha)T_\alpha^{-1}u + P_h^\alpha F(\lambda, u),$$

where I is the identity of $W^{-1,\infty}$.

Lemma 6.2 (Rheinboldt [11, Lemma 5.1]). *The operator $\overline{F}_h^\alpha$ satisfies the following*:

(1) $\overline{F}_h^\alpha(\lambda, u) = 0$ *for some* $(\lambda, u) \in \Lambda \times W_0^{1,\infty}$ *if and only if* $(\lambda, u) \in \Lambda \times \mathring{S}_h$ *and* $F_h(\lambda, u) = 0$.

(2) $\overline{F}_h^\alpha$ *is a Fredholm operator of index* 1 *on* $\Lambda \times W_0^{1,\infty}$.

By Lemma 6.2, we have the following theorem as a consequence of the Fink–Rheinboldt theory.

Theorem 6.3. *Suppose that F is C^d mapping ($d \geqslant 1$). Then the set of the finite elements solutions of Problem* 6.1^*_{FE},

$$\mathscr{M}_h := \{(\lambda_h, u_h) \in \mathscr{R}(F_h, \Lambda \times \mathring{S}_h)) \mid F_h(\lambda_h, u_h) = 0\},$$

is a C^d manifold without boundary.

7. A priori error estimates of the FE solution manifold: Regular branches

We are ready to start to consider a priori error estimates of the FE solution manifold $\mathscr{M}_h$. In the consideration of error estimates, we always assume the following.

Assumption 7.1. *We assume that*

(1) *Assumption* 3.1 *with d (i.e., F is a C^d Fredholm map).*

(2) $0 \in F(\mathscr{R}(F, \mathscr{S}))$ *(i.e.,* $\mathscr{M}_0 \neq \emptyset$*).*

(3) *Assumption* 5.1 *(i.e.,* $u \in W^{2,p^*}$, $2 \leqslant p^* \leqslant \infty$ *for any* $(\lambda, u) \in \mathscr{M}_0$*).*

(4) *The triangulation of $\mathring{S}_h$ (in one-dimensional case, the partition of J into small intervals) is regular* [6, p. 124] *and* $\lim_{h\to 0} \inf_{v_h \in \mathring{S}_h} \|u - v_h\|_{H_0^1} = 0$, *for any* $u \in H_0^1$.

(5) *The triangulation of $\mathring{S}_h$ satisfies the inverse assumption* [6, p. 140].

In the sequel, we denote by $\widehat{\Pi}_h : W_0^{1,1} \subset C^0 \to \mathring{S}_h$ the interpolant projection defined in [6, Theorem 3.1.4]. We also denote by C or C_i, i is nonnegative integers, generic constants which are independent of $h > 0$.

The main tool of our a priori error estimates is the following implicit function theorem due to Brezzi et al. [3, Theorem 1].

Theorem 7.2. *Let X, Y and Z be Banach spaces. Let $S \subset X$ and $y : S \to Y$ a function defined in S. Let f be a C^1 mapping defined in a neighborhood of $S \times y(S)$. Suppose that the function $S \ni x \mapsto y(x) \in Y$ satisfies the uniform Lipschitz condition; there exists a constant C_0 such that*

$$\|y(x) - y(x^*)\|_Y \leqslant C_0 \|x - x^*\|_X, \quad \forall x, x^* \in S.$$

Suppose in addition that the following hypotheses hold:

(i) *for all $x_0 \in S$, $D_y f(x_0, y(x_0))$ is an isomorphism of Y onto Z with*

$$\sup_{x_0 \in S} \|(D_y f(x_0, y(x_0)))^{-1}\|_{\mathscr{L}(Z,Y)} \leqslant C_1,$$

(ii) *we have*

$$\sup_{x_0 \in S} \|D_x f(x_0, y(x_0))\|_{\mathscr{L}(X,Z)} \leqslant C_2,$$

and there exists a monotonically increasing function $L_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x_0 \in S$ and all $(x, y) \in B_\xi((x_0, y(x_0)))$

$$\|Df(x, y) - Df(x_0, y(x_0))\|_{\mathscr{L}(X \times Y, Z)} \leqslant L_1(\xi)(\|x - x_0\|_X + \|y - y(x_0)\|_Y).$$

Then, one can find three constants $a, b, d > 0$ depending only on C_0, C_1, C_2 and L_1 such that, under the condition

$$\sup_{x_0 \in S} \|f(x_0, y(x_0))\|_Z \leqslant d,$$

there exists a unique C^1 function $g : \bigcup_{x_0 \in S} B_a(x_0) \to Y$ which satisfies

$$f(x, g(x)) = 0,$$

and maps $B_a(x_0)$ into $B_b(y(x_0))$ for $x_0 \in S$. Moreover, we have for all $x_0 \in S$ and all $x \in B_a(x_0)$

$$\|g(x) - y(x_0)\|_Y \leqslant K_0(\|x - x_0\|_X + \|f(x_0, y(x_0))\|_Z),$$

where the constant $K_0 > 0$ depends only on C_1, C_2.

Our first main theorem is as follows.

Theorem 7.3. *Suppose that Assumption 7.1 holds for $d \geqslant 2$. Also suppose that, at $(\lambda_0, u_0) \in \mathscr{M}_0$, $D_u F(\lambda_0, u_0) \in \mathscr{L}(W_0^{1,\infty}, W^{-1,\infty})$ is an isomorphism.*

Then, for sufficiently small $h > 0$, there exist positive constants ε_{λ_0}, $K_0(\lambda_0)$, and a unique C^2 map $[\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0}] \ni \lambda \mapsto \tilde{u}_h(\lambda) \in \mathring{S}_h$ such that

$$F_h(\lambda, \tilde{u}_h(\lambda)) = 0, \tag{7.1}$$

$$\|\tilde{u}_h(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{H_0^1} \leqslant K_0(\lambda_0) \|u(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{H_0^1}. \tag{7.2}$$

Moreover, we have for some constant $K_1(\lambda_0) > 0$

$$\|\tilde{u}_h(\lambda) - u(\lambda)\|_{H_0^1} \leqslant K_1(\lambda_0)\|u(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{H_0^1}. \tag{7.3}$$

Here, the constants $K_0(\lambda_0)$ *and* $K_1(\lambda_0)$ *are independent of* h *and* $\lambda \in [\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0}]$.

Proof. Since the proof is somewhat long, we divide it into several steps.

Step 1: It follows from Lemma 3.2 that, for $(\lambda, u) \in \mathscr{M}_0$, $DF(\lambda, u) \in \mathscr{L}(\mathbb{R} \times H_0^1, H^{-1})$ and

$$\begin{aligned} &\Lambda \times W_0^{1,\infty} \ni (\lambda, u) \mapsto DF(\lambda, u) \in \mathscr{L}(\mathbb{R} \times H_0^1, H^{-1}) \text{ is} \\ &\text{Lipschitz continuous on bounded subset.} \end{aligned} \tag{7.4}$$

We, moreover, claim that, if $D_uF(\lambda, u) \in \mathscr{L}(W_0^{1,\infty}, W^{-1,\infty})$ is an isomorphism for $(\lambda, u) \in \mathscr{S}$, $D_uF(\lambda, u)$ is an isomorphism from H_0^1 to H^{-1} as well.

Define $Q, R \in \mathscr{L}(H_0^1, H^{-1})$ by

$$_2\langle Q\psi, v\rangle_2 := \int_J \alpha(x)\psi' v' \,\mathrm{d}x, \quad _2\langle R\psi, v\rangle_2 := \int_J \beta(x)\psi v \,\mathrm{d}x, \quad \forall \psi, v \in H_0^1,$$

where $\alpha(x) := a_y(\lambda, x, u'(x))$ and $\beta(x) := f_y(\lambda, x, u(x))$.

By Theorem 4.3, Q is a Fredholm operator of index 0 and R is compact. Hence, $D_uF(\lambda, u) = Q + R \in \mathscr{L}(H_0^1, H^{-1})$ is a Fredholm operator of index 0. Therefore, if $\operatorname{Ker} D_uF(\lambda, u) \subset H_0^1$ is trivial, $D_uF(\lambda, u) \in \mathscr{L}(H_0^1, H^{-1})$ is an isomorphism.

Let $\psi \in H_0^1$ be such that $D_uF(\lambda, u)\psi = 0$. This means that $-(\alpha(x)\psi')' + \beta(x)\psi = 0$ in the distributional sense. Since $\beta \in L^1$, we conclude $\psi \in W^{2,1}$ by a standard regularity argument. Hence, $\psi \in W_0^{1,\infty}$ and $0 = D_uF(\lambda, u)\psi \in W^{-1,\infty}$. Since we assumed that $D_uF(\lambda, u) \in \mathscr{L}(W_0^{1,\infty}, W^{-1,\infty})$ is an isomorphism, we obtain $\psi = 0$. Therefore, our claim is proved.

Step 2: We prepare inequalities which we will use later. By Assumption 7.1(5) we have the inverse inequality [6, Theorem 3.2.6],

$$\|v_h\|_{W_0^{1,\infty}} \leqslant C_3 h^{-1/2} \|v_h\|_{H_0^1}, \qquad v_h \in \mathring{S}_h. \tag{7.5}$$

Since $D_uF(\lambda_0, u_0) \in \mathscr{L}(W_0^{1,\infty}, W^{-1,\infty})$ is an isomorphism, we conclude by the implicit function theorem that there exist $\varepsilon_1 > 0$ and a unique C^2 map

$$(\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_1) \ni \lambda \mapsto u(\lambda) \in W_0^{1,\infty}$$

such that $u_0 = u(\lambda_0)$ and $F(\lambda, u(\lambda)) = 0$. Thus, it follows that

$$\|\widehat{\Pi}_h u(\lambda^*) - \widehat{\Pi}_h u(\lambda)\|_{H_0^1} \leqslant C_4 |\lambda^* - \lambda|, \tag{7.6}$$

for all $\lambda, \lambda^* \in (\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_1)$. In (7.6) we used the fact that $\sup_{h>0} \|\widehat{\Pi}_h\|_{\mathscr{L}(H_0^1, H_0^1)} < \infty$ (see, e.g., [6, Theorem 3.1.6]).

By Theorem 5.3 we know that $u(\lambda) \in W^{2,p^*}$ and

$$C_5 := \sup\left\{ \|u(\lambda)\|_{W^{2,p^*}};\ \lambda \in \left[\lambda_0 - \frac{\varepsilon_1}{2}, \lambda_0 + \frac{\varepsilon_1}{2}\right] \right\} < \infty. \tag{7.7}$$

Thus, by [6, Theorem 3.1.6], we see

$$\|u(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{W_0^{1,\infty}} \leqslant CC_5 h^{1-1/p^*}. \tag{7.8}$$

Let $\alpha_0(x) := a_y(\lambda_0, x, u_0'(x))$. Since $a_y(\lambda_0, x, u_0'(x))^{-1} \in L^\infty$ $((\lambda_0, u_0) \in \mathscr{M}_0 \subset \mathscr{S})$, $u_0 \in W^{2,p^*} \subset C^1[0,1]$, and Assumption 5.1(1), we can assume, without loss of generality, that

$$a_y(\lambda_0, x, u_0'(x)) \geqslant \exists \delta_0 > 0. \tag{7.9}$$

Then, we define the bilinear form $A : H_0^1 \times H_0^1 \to \mathbb{R}$ by

$$A(u,v) := \int_J \alpha_0(x) u' v' \, \mathrm{d}x, \quad \forall u, v \in H_0^1.$$

Also, we define the canonical projection $\Pi_h^0 : H_0^1 \to \mathring{S}_h$ by

$$A(u - \Pi_h^0 u, v_h) = 0, \quad \forall v_h \in \mathring{S}_h, \tag{7.10}$$

for $u \in H_0^1$. Then, it follows from Assumption 7.1(4) and (7.9) that

$$\lim_{h \to 0} \|u - \Pi_h^0 u\|_{H_0^1} = 0, \quad \forall u \in H_0^1. \tag{7.11}$$

Now, define $T_0 : H^{-1} \to H_0^1$ by

$$_2\langle \Phi, v \rangle_2 = \int_J \alpha_0(x) (T_0 \Phi)' v' \, \mathrm{d}x, \quad \forall v \in H_0^1,$$

for $\Phi \in H^{-1}$, and define $\overline{F}_h^0 : \Lambda \times W_0^{1,\infty} \to W^{-1,\infty}$ by

$$\overline{F}_h^0(\lambda, u) := (I - P_h^0) T_0^{-1} u + P_h^0 F(\lambda, u),$$

where I is the identity map of $W_0^{-1,\infty}$ and $P_h^0 := T_0^{-1} \Pi_h^0 T_0 \in \mathscr{L}(W^{-1,\infty}, W^{-1,\infty})$.

Note that, by Lemma 6.2, $\overline{F}_h^0(\lambda, u) = 0$ if and only if $(\lambda, u) \in \mathscr{M}_h$.

By the definition of Π_h^0 and P_h^0, we immediately get

$$C_6 := \sup_{h > 0} \|P_h^0\|_{\mathscr{L}(H^{-1}, H^{-1})} < \infty. \tag{7.12}$$

It follows from (7.7), (7.12), and [6, Theorem 3.1.6] that

$$\begin{aligned} \|\overline{F}_h^0(\lambda, \widehat{\Pi}_h u(\lambda))\|_{H^{-1}} &\leqslant C_6 \|F(\lambda, u(\lambda)) - F(\lambda, \widehat{\Pi}_h u(\lambda))\|_{H^{-1}} \\ &\leqslant C_6 \left(\int_0^1 \|\Psi_t\|_{\mathscr{L}(H_0^1, H^{-1})} \mathrm{d}t \right) \|u(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{H_0^1} \\ &\leqslant C_7 h, \qquad \forall \lambda \in \left[\lambda_0 - \tfrac{1}{2}\varepsilon_1, \lambda_0 + \tfrac{1}{2}\varepsilon_1\right], \end{aligned} \tag{7.13}$$

where $\Psi_t := D_u F(\lambda, (1-t) u(\lambda) + t \widehat{\Pi}_h u(\lambda)) \in \mathscr{L}(H_0^1, H^{-1})$. Therefore, we obtain

$$\lim_{h \to 0} \sup \left\{ h^{-\frac{1}{2}} \|\overline{F}_h^0(\lambda, \widehat{\Pi}_h u(\lambda))\|_{H^{-1}} ; \lambda \in \left[\lambda_0 - \tfrac{1}{2}\varepsilon_1, \lambda_0 + \tfrac{1}{2}\varepsilon_1\right] \right\} \leqslant \lim_{h \to 0} C_7 h^{1/2} = 0. \tag{7.14}$$

Step 3: We claim that there exist a positive $\varepsilon_2 > 0$ and a constant $C_8 > 0$ independent of $h > 0$ and $\lambda \in [\lambda_0 - \varepsilon_2, \lambda_0 + \varepsilon_2]$ with

$$\|D_u \overline{F}_h^0(\lambda, \widehat{\Pi}_h u(\lambda))v_h\|_{H^{-1}} \geqslant C_8 \|v_h\|_{H_0^1}, \quad \forall v_h \in \mathring{S}_h. \tag{7.15}$$

First, we note that, by (7.4) and Step 1, the mapping

$$(\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_1) \ni \lambda \mapsto (D_u F(\lambda, u(\lambda)))^{-1} \in \mathscr{L}(H^{-1}, H_0^1)$$

is continuous. Thus, we set

$$\omega := \max_{\lambda \in [\lambda_0 - \frac{1}{2}\varepsilon_1, \lambda_0 + \frac{1}{2}\varepsilon_1]} \|(D_u F(\lambda, u(\lambda)))^{-1}\|_{\mathscr{L}(H^{-1}, H_0^1)}.$$

Next, we write

$$\begin{aligned} D_u \overline{F}_h^0(\lambda, \widehat{\Pi}_h u(\lambda))v_h = {} & D_u F(\lambda, u(\lambda))v_h^{(a)} \\ & + P_h^0(D_u F(\lambda, \widehat{\Pi}_h u(\lambda)) - D_u F(\lambda, u(\lambda)))v_h^{(b)} \\ & - (I - P_h^0)(-T_0^{-1} + D_u F(\lambda_0, u_0))v_h^{(c)} \\ & + (I - P_h^0)(D_u F(\lambda_0, u_0) - D_u F(\lambda, u(\lambda)))v_h^{(d)}. \end{aligned} \tag{7.16}$$

Let us examine every term of (7.16). For (a), we immediately see

$$\|D_u F(\lambda, u(\lambda))v_h\|_{H^{-1}} \geqslant \omega^{-1} \|v_h\|_{H_0^1}. \tag{7.17a}$$

For (b), it follows from (7.4), (7.8), and (7.12) that

$$\begin{aligned} & \|P_h^0(D_u F(\lambda, \widehat{\Pi}_h u(\lambda)) - D_u F(\lambda, u(\lambda)))v_h\|_{H^{-1}} \\ & \quad \leqslant C_9 \|\widehat{\Pi}_h u(\lambda) - u(\lambda)\|_{W_0^{1,\infty}} \|v_h\|_{H_0^1} \leqslant C C_5 C_9 h^{1-1/p^*} \|v_h\|_{H_0^1}. \end{aligned} \tag{7.17b}$$

For (c), we remind that

$$_2\langle(-T_0^{-1} + D_u F(\lambda_0, u_0))\psi, v\rangle_2 = \int_J f_y(\lambda_0, x, u_0(x))\psi\, v \,\mathrm{d}x,$$

and $-T_0^{-1} + D_u F(\lambda_0, u_0) \in \mathscr{L}(H_0^1, H^{-1})$ is compact. Therefore, we conclude that

$$\lim_{h \to 0} \|(I - P_h^0)(-T_0^{-1} + D_u F(\lambda_0, u_0))\|_{\mathscr{L}(H_0^1, H^{-1})} = 0 \tag{7.17c}$$

because of (7.11).

For (d) we observe the following. By (7.4) there exists a constant C_{10} such that

$$\|D_u F(\lambda^*, u(\lambda^*)) - D_u F(\lambda, u(\lambda))\|_{\mathscr{L}(H_0^1, H^{-1})} \leqslant C_{10}|\lambda^* - \lambda|.$$

Take $\varepsilon_2 > 0$ so that $\varepsilon_2^{-1} \geqslant 2\omega C_{10} \sup_{h>0} \|I - P_h^0\|_{\mathscr{L}(H^{-1}, H^{-1})}$. Then we have

$$\|(I - P_h^0)(D_u F(\lambda_0, u_0) - D_u F(\lambda, u(\lambda)))\|_{\mathscr{L}(H_0^1, H^{-1})} \leqslant \tfrac{1}{2}\omega^{-1}, \tag{7.17d}$$

for any $\lambda \in [\lambda_0 - \varepsilon_2, \lambda_0 + \varepsilon_2]$.

From (7.16) and (7.17), we obtain

$$\|D_u\overline{F}_h^0(\lambda,\widehat{\Pi}_h u(\lambda))v_h\|_{H^{-1}} \geqslant \left(\tfrac{1}{2}\omega^{-1} - \delta(h)\right)\|v_h\|_{H_0^1},$$

with $\lim_{h\to 0}\delta(h) = 0$. Therefore, we prove the claim (7.15) for sufficiently small h.

Step 4: Again, we prepare a few inequalities. It follows from (7.8) that

$$\|D_\lambda\overline{F}_h^0(\lambda,\widehat{\Pi}_h u(\lambda))\|_{H^{-1}} \leqslant C_6\left(\|D_\lambda F(\lambda,u(\lambda))\|_{H^{-1}} + C_{11}h^{1-1/p^*}\right) \leqslant C_{12}, \tag{7.18}$$

for all $\lambda\in[\lambda_0-\varepsilon_1/2,\lambda_0+\varepsilon_1/2]$ and $h>0$.

By (7.4) and (7.12) we have

$$\begin{aligned}&\|D\overline{F}_h^0(\lambda^*,v_h^*) - D\overline{F}_h^0(\lambda,\widehat{\Pi}_h u(\lambda))\|_{\mathscr{L}(\mathbb{R}\times H_0^1,H^{-1})}\\ &\quad\leqslant C_{13}(|\lambda^*-\lambda| + \|v_h^* - \widehat{\Pi}_h u(\lambda)\|_{W_0^{1,\infty}}),\quad \forall v_h\in\overset{\circ}{S}_h,\end{aligned}$$

where $C_{13}=C_{13}(|\lambda|,|\lambda^*|,\|v_h^*\|_{W_0^{1,\infty}})$ is independent of h. Hence, by (7.5), there exists a monotonically increasing function $L_1:\mathbb{R}^+\to\mathbb{R}^+$ independent of h such that, for all $\lambda,\lambda^*\in[\lambda_0-\varepsilon_1/2,\lambda_0+\varepsilon_1/2]$ and all $v_h^*\in\overset{\circ}{S}_h$ with

$$h^{-1/2}(|\lambda^*-\lambda| + \|v_h^* - \widehat{\Pi}_h u(\lambda)\|_{H_0^1}) \leqslant \xi,$$

we have

$$\begin{aligned}&\|D\overline{F}_h^0(\lambda^*,v_h^*) - D\overline{F}_h^0(\lambda,\widehat{\Pi}_h u(\lambda))\|_{\mathscr{L}(\mathbb{R}\times H_0^1,H^{-1})}\\ &\quad\leqslant L_1(\xi)h^{-1/2}(|\lambda^*-\lambda| + \|v_h^* - \widehat{\Pi}_h u(\lambda)\|_{H_0^1}).\end{aligned} \tag{7.19}$$

Step 5: This is the final stage of the proof. By (7.6), (7.14), (7.15), (7.18) and (7.19) we can apply Theorem 7.2 to the operator $\overline{F}_h^0$ in the following situation:

$$\begin{aligned}&X=\mathbb{R}\text{ with norm } h^{-1/2}|\lambda|,\\ &Y=\overset{\circ}{S}_h\subset H_0^1\text{ with norm } h^{-1/2}\|v_h\|_{H_0^1},\\ &Z=\overset{\circ}{S}_h\subset H^{-1}\text{ with norm } h^{-1/2}\|T_0^{-1}v_h\|_{H^{-1}},\\ &S=[\lambda_0-\varepsilon_{\lambda_0},\lambda_0+\varepsilon_{\lambda_0}]\text{ with } \varepsilon_{\lambda_0}:=\min\left(\tfrac{1}{2}\varepsilon_1,\varepsilon_2\right),\\ &y(\lambda)=\widehat{\Pi}_h u(\lambda).\end{aligned}$$

Since $\|A_h\|_{\mathscr{L}(X\times Y,Z)} = \|A_h\|_{\mathscr{L}(\mathbb{R}\times H_0^1,H^{-1})}$ for all $A_h\in\mathscr{L}(\mathbb{R}\times\overset{\circ}{S}_h,\overset{\circ}{S}_h)$ and (7.14), there exists a unique C^2 function $[\lambda_0-\varepsilon_{\lambda_0},\lambda_0+\varepsilon_{\lambda_0}]\ni\lambda\mapsto\tilde{u}_h(\lambda)\in\overset{\circ}{S}_h$ such that

$$\overline{F}_h^0(\lambda,\tilde{u}_h(\lambda)) = 0, \tag{7.20}$$

and the inequality

$$\|\tilde{u}_h(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{H_0^1} \leqslant C_{14}\|\overline{F}_h^0(\lambda,\widehat{\Pi}_h u(\lambda))\|_{H^{-1}} \tag{7.21}$$

holds for all $\lambda \in [\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0}]$. From Lemma 6.2, we get (7.1) immediately from (7.20). Combining (7.13) and (7.21), we obtain (7.2), (7.3), and complete the proof of Theorem 7.3. □

Corollary 7.4. *Suppose that the assumptions of Theorem 7.3 hold. Then, there exists a constant $K_2(\lambda_0) > 0$ independent of $h > 0$ and $\lambda \in [\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0}]$ such that*

$$\|u(\lambda) - \tilde{u}_h(\lambda)\|_{W_0^{1,\infty}} \leqslant K_2(\lambda_0) h^{1/2}, \tag{7.22}$$

for any $\lambda \in [\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0}]$ and sufficiently small $h > 0$.

Proof. By (7.2) and the inverse inequality [6, Theorem 3.2.6], we have

$$\|\tilde{u}_h(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{W_0^{1,\infty}} \leqslant K_1(\lambda_0) h^{1/2},$$

for all $\lambda \in [\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0}]$. It follows from (7.8) that

$$\begin{aligned}\|u(\lambda) - \tilde{u}_h(\lambda)\|_{W_0^{1,\infty}} &\leqslant \|u(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{W_0^{1,\infty}} + \|\tilde{u}_h(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{W_0^{1,\infty}} \\ &\leqslant C C_5 h^{1-(1/p^*)} + K_1(\lambda_0) h^{1/2} \leqslant \exists K_2(\lambda_0) h^{1/2}. \qquad \square\end{aligned}$$

Theorem 7.5. *Suppose that Assumption 7.1 holds for $d \geqslant 2$. Also, suppose that $\widetilde{\mathscr{M}}_0 \subseteq \mathscr{M}_0$ is a compact regular branch, that is, there is a compact interval $\widetilde{\Lambda} \subset \Lambda$ and C^2 map $\widetilde{\Lambda} \ni \lambda \mapsto u(\lambda) \in W_0^{1,\infty}$ such that*

$$\widetilde{\mathscr{M}}_0 = \{(\lambda, u(\lambda)) \in \mathscr{M}_0 \mid D_u F(\lambda, u(\lambda)) \text{ is an isomorphism for } \forall \lambda \in \widetilde{\Lambda}\}.$$

Then, for sufficiently small $h > 0$, there exists the corresponding finite element solution branch $\widetilde{\mathscr{M}}_h \subset \mathscr{M}_h$ which is parametrized by the same $\lambda \in \widetilde{\Lambda}$ and

$$\|\widehat{\Pi}_h u(\lambda) - u_h(\lambda)\|_{H_0^1} \leqslant K_3 \|u(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{H_0^1}, \tag{7.23}$$

$$\|u(\lambda) - u_h(\lambda)\|_{H_0^1} \leqslant K_4 \|u(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{H_0^1}, \tag{7.24}$$

$$\|u(\lambda) - u_h(\lambda)\|_{W_0^{1,\infty}} \leqslant K_5 h^{1/2} \tag{7.25}$$

for all $\lambda \in \widetilde{\Lambda}$, $u(\lambda) \in \widetilde{\mathscr{M}}_0$, $u_h(\lambda) \in \widetilde{\mathscr{M}}_h$. Here, $K_3, K_4, K_5 > 0$ are constants independent of h and λ.

Moreover, we have

$$\widetilde{\mathscr{M}}_h \subset \mathscr{R}(F, \mathscr{S}). \tag{7.26}$$

Proof. From Theorem 7.3 and Corollary 7.4, (7.23)–(7.25) are obtained immediately.

To show (7.26) we just have to realize that $D_u F(\lambda, u(\lambda)) \in \mathscr{L}(W_0^{1,\infty}, W^{-1,\infty})$ is an isomorphism for each $\lambda \in \widetilde{\Lambda}$ and

$$D_u F(\lambda, u_h(\lambda)) = D_u F(\lambda, u(\lambda)) + B_h,$$

where $B_h := D_u F(\lambda, u_h(\lambda)) - D_u F(\lambda, u(\lambda))$ and $\|B_h\|_{\mathscr{L}(W_0^{1,\infty}, W^{-1,\infty})} \to 0$ as $h \to 0$ because of (7.25). □

Remark 7.6. (1) For the estimates (7.2), (7.3), (7.23), and (7.24), we have the following inequality [6, Theorem 3.1.6]:

$$\|u(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{H_0^1} \leqslant Ch|u(\lambda)|_{H^2}.$$

(2) In linear cases, with certain assumptions of regularity of solutions, we would have error estimates like

$$\|\widehat{\Pi}_h u - u_h\|_{H_0^1} \leqslant Ch^2,$$

(see [15, Section 30]). It is not very clear whether or not the convergence rate of (7.2) and (7.23) is optimal. We might be able to improve the convergence rate with further assumptions for the regularity of solutions (Assumption 5.1 might not be enough to improve (7.2), (7.23)).

For $\|\cdot\|_{W_0^{1,p^*}}$-estimate, we have the following. Suppose that we have

$$\lim_{h\to 0} \|u - \Pi_h^0 u\|_{W_0^{1,p^*}} = 0, \quad \forall u \in W_0^{1,p^*}, \tag{7.27}$$

where $\Pi_h^0 \in \mathscr{L}(W_0^{1,p^*}, W_0^{1,p^*})$ is defined by (7.10). For example, we can show that (7.27) is true for piecewise linear elements.

Theorem 7.7. *Suppose that assumptions of Theorem 7.5 and (7.27) hold. Then, for sufficiently small $h > 0$, for the corresponding finite element solution branch $\widetilde{\mathscr{M}}_h \subset \mathscr{M}_h$, we have the following estimates*:

$$\|\widehat{\Pi}_h u(\lambda) - u_h(\lambda)\|_{W_0^{1,p^*}} \leqslant K_6 \|u(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{W_0^{1,p^*}},$$

$$\|u(\lambda) - u_h(\lambda)\|_{W_0^{1,p^*}} \leqslant K_7 \|u(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{W_0^{1,p^*}},$$

$$\|u(\lambda) - u_h(\lambda)\|_{W_0^{1,\infty}} \leqslant K_8 h^{1-1/p^*},$$

for all $\lambda \in \widetilde{\Lambda}$, $u(\lambda) \in \widetilde{\mathscr{M}}_0$, $u_h(\lambda) \in \widetilde{\mathscr{M}}_h$. Here, $K_6, K_7, K_8 > 0$ are constants independent of h and λ.

Remark 7.8. Similarly to linear cases, it follows from Theorem 7.7 that

$$\|u(\lambda) - u_h(\lambda)\|_{W_0^{1,\infty}} \leqslant Ch,$$

if $p^* = \infty$, that is, $u(\lambda) \in W^{2,\infty}$.

8. A priori error estimates of the FE solution manifold: Around turning points

Let us consider a priori error estimates around turning points and/or on "steep slopes". Basic idea is as follows: just rotate the coordinate 90° and do exactly the same thing as in Section 7.

Recall that by the argument in Section 4 we know that we have either Case 1 or Case 2 for $(\lambda, u) \in \mathscr{M}_0 \subset \mathscr{R}(F, S)$;

Case 1: $\mathrm{Ker}\, D_uF(\lambda,u)=\{0\}$ and $D_\lambda F(\lambda,u)\in \mathrm{Im}\, D_uF(\lambda,u)$.

Case 2: $\dim \mathrm{Ker}\, D_uF(\lambda,u)=1$ and $D_\lambda F(\lambda,u)\notin \mathrm{Im}\, D_uF(\lambda,u)$.

Suppose that $\gamma\in\mathbb{R}$ and $x_0\in J$ are given in a certain way and fixed. Define $G:\mathscr{R}(F,\mathscr{S})\to \mathbb{R}\times W^{-1,\infty}$ by

$$G(\lambda,u):=(u(x_0)-\gamma,F(\lambda,u))$$

for $(\lambda,u)\in\mathscr{R}(F,\mathscr{S})$. Then, we have

$$DG(\lambda,u)(\mu,\psi)=(\psi(x_0),\mu D_\lambda F(\lambda,u)+D_uF(\lambda,u)\psi), \tag{8.1}$$

for $\mu\in\mathbb{R}$ and $\psi\in W_0^{1,\infty}$. First, we prepare the following lemma.

Lemma 8.1. *Let $(\lambda,u)\in\mathscr{R}(F,\mathscr{S})$ and (μ_0,ψ_0) the basis of* $\mathrm{Ker}\, DF(\lambda,u)$. *Suppose that $D_\lambda F(\lambda,u)\neq 0$. Then, we have $\|\psi_0\|_{C^0}>0$. Moreover, with $x_0\in J$ such that $\psi_0(x_0)\neq 0$, $DG(\lambda,u)$ is an isomorphism of $\mathbb{R}\times W_0^{1,\infty}$ to $\mathbb{R}\times W^{-1,\infty}$ for any $\gamma\in\mathbb{R}$.*

Proof. Suppose that we are in Case 1. It follows from $D_\lambda F(\lambda,u)\neq 0$ that

$$\psi_1:=-D_uF(\lambda,u)^{-1}(D_\lambda F(\lambda,u))\neq 0.$$

Thus, the basis of $\mathrm{Ker}\, DF(\lambda,u)$ should be written as $(\mu_0,\mu_0\psi_1)$. Hence, we obtain $\|\psi_0\|_{C^0}=|\mu_0|\|\psi_1\|_{C^0}>0$.

Suppose that we are in Case 2. Let $\psi_0\in W_0^{1,\infty}$ be such that $\mathrm{Ker}\, D_uF(\lambda,u)=\mathrm{span}\{\psi_0\}$. Then, $(0,\psi_0)$ is the basis of $\mathrm{Ker}\, DF(\lambda,u)$ and the first part of Lemma 8.1 is trivial in this case.

Now, let us consider $DG(\lambda,u)$. Let $(\mu,\psi)\in\mathrm{Ker}\, DG(\lambda,u)$, and $x_0\in J$ such that $\psi(x_0)\neq 0$. Then, there exists $\delta\in\mathbb{R}$ such that $(\mu,\psi)=\delta(\mu_0,\psi_0)$. Thus, it follows from (8.1) and $\psi_0(x_0)\neq 0$ that $(\mu,\psi)=(0,0)$. Hence, $DG(\lambda,u)$ is one-to-one.

Let (z,Φ) be any element of $\mathbb{R}\times W^{-1,\infty}$. Since $DF(\lambda,u)$ is onto, there exists $(\rho,\phi)\in\mathbb{R}\times W_0^{1,\infty}$ such that $\Phi=DF(\lambda,u)(\rho,\phi)$. Hence, we obtain $(z,\Phi)=DG(\lambda,u)((\rho,\phi)+\delta(\mu_0,\psi_0))$, where $\delta:=(z-\phi(x_0))/\psi_0(x_0)$. Therefore, we have showed that $DG(\lambda,u)$ is onto and an isomorphism. □

From Lemma 8.1, we immediately obtain the following corollary.

Corollary 8.2. *Let $(\lambda,u)\in\mathscr{R}(F,\mathscr{S})$. Suppose that $D_\lambda F(\lambda,u)\neq 0$. Then, for sufficiently small $h>0$, there exists a nodal point $x_0\in J$ of $\mathring{S}_h$ such that $DG(\lambda,u)$ is an isomorphism.*

Remark 8.3. In Lemma 8.2 we showed that we always can choose a nodal point of $\mathring{S}_h$ so that $DG(\lambda,u)$ is an isomorphism if $D_\lambda F(\lambda,u)\neq 0$. For example, if a nodal point $x_0\in J$ is taken so that $\psi_0(x_0)$ is nearly equal to $\|\psi_0\|_{C^0}$, then $DG(\lambda,u)$ is an isomorphism.

Indeed, the manner of PITCON, a continuation program developed by Rheinboldt and his colleagues [11], of choosing the continuation index is consistent to the above fact. After getting a point $(\lambda_h,u_h)\in\mathscr{M}_h$, PITCON computes the tangent vector $t_h=(y_0,\ldots,y_k)\in T_{(\lambda_h,u_h)}\mathscr{M}_h$ of the solution manifold $\mathscr{M}_h$ (remember that $\mathrm{Ker}\, DF_h(\lambda_h,u_h)=T_{(\lambda_h,u_h)}\mathscr{M}_h$). Then, the continuation index i_c is taken so that $|y_{i_c}|=\|t_h\|_\infty$. In our case, $(y_0,\ldots,y_k)$ is like $(\mu_{0h},\psi_{0h}(x_1),\ldots,\psi_{0h}(x_k))$, where $x_1,\ldots,x_k$ are nodal points of $\mathring{S}_h$, (μ_{0h}, ψ_{0h}) is the basis of $\operatorname{Ker} DF_h(\lambda_h, u_h)$, and $y_{i_c} = \psi_{0h}(x_0)$ (see [11] for the detail). Thus, for sufficiently small $h > 0$, $|\psi_{0h}(x_0)|$ would be very close to $\|\psi_0\|_{C^0}$, and $\psi_0(x_0)$ is not zero.

Hence, in practical computation, we may expect that PITCON takes the right nodal point x_0, and $DG(\lambda, u) \in \mathscr{L}(\mathbb{R} \times W_0^{1,\infty}, \mathbb{R} \times W^{-1,\infty})$ is an isomorphism.

Suppose that $D_\lambda F(\lambda, u) \neq 0$ at $(\lambda, u) \in \mathscr{M}_0$. To "rotate" the coordinate we define the operator $H : \mathbb{R} \times \mathscr{S} \to \mathbb{R} \times W^{-1,\infty}$ by

$$H(\gamma, \lambda, u) := (u(x_0) - \gamma, F(\lambda, u)), \quad \gamma \in \mathbb{R}, (\lambda, u) \in \mathscr{S},$$

where $x_0 \in J$ is taken so that $D_{(\lambda,u)}H(\gamma, \lambda, u) = DG(\lambda, u) \in \mathscr{L}(\mathbb{R} \times W_0^{1,\infty}, \mathbb{R} \times W^{-1,\infty})$ is an isomorphism.

Note that

$$DH(\gamma, \lambda, u)(s, t, \psi) = (-s + \psi(x_0), DF(\lambda, u)(t, \psi)) \tag{8.2}$$

for $(s, t) \in \mathbb{R}^2$ and $\psi \in W_0^{1,\infty}$. Also, note that, by the implicit function theorem, for each $(\lambda, u) \in \mathscr{M}_0$ such that $D_\lambda F(\lambda, u) \neq 0$, there exist $\varepsilon_0 > 0$ and a unique C^2 map $(u(x_0) - \varepsilon_0, u(x_0) + \varepsilon_0) \ni \gamma \mapsto (\lambda(\gamma), u(\gamma)) \in \mathscr{M}_0$ such that $(\lambda, u) = (\lambda(\gamma_0), u(\gamma_0))$ with $\gamma_0 := u(x_0)$, and $H(\gamma, \lambda(\gamma), u(\gamma)) = (0, 0)$, that is, $F(\lambda(\gamma), u(\gamma)) = 0$ and $u(\gamma)(x_0) = \gamma$ for any γ.

Suppose that $D_\lambda F(\lambda_0, u_0) \neq 0$ at $(\lambda_0, u_0) \in \mathscr{M}_0$. Then by Corollary 8.2 there exists a nodal point $x_0 \in J$ of $\mathring{S}_h$ such that $D_{(\lambda,u)}H(\gamma, \lambda_0, u_0) = DG(\lambda_0, u_0)$ is an isomorphism of $\mathbb{R} \times W_0^{1,\infty}$ to $\mathbb{R} \times W^{-1,\infty}$ for sufficiently small $h > 0$.

Theorem 8.4. *Suppose that Assumption 7.1 holds for $d \geqslant 2$. Let $D_\lambda F(\lambda_0, u_0) \neq 0$ at $(\lambda_0, u_0) \in \mathscr{M}_0$. We assume, without loss of generality, that there exists $x_0 \in J$ such that x_0 is a nodal point of $\mathring{S}_h$ for all sufficiently small $h > 0$ and $D_{(\lambda,u)}H(\gamma, \lambda_0, u_0)$ is an isomorphism.*

Then, there exist positive constants ε_0, $K_0(\lambda_0, u_0)$, $K_1(\lambda_0, u_0)$, and a unique C^2 map $[u_0(x_0) - \varepsilon_0, u_0(x_0) + \varepsilon_0] \ni \gamma \mapsto (\tilde{\lambda}_h(\gamma), \tilde{u}_h(\gamma)) \in \Lambda \times \mathring{S}_h$ such that

$$F_h(\tilde{\lambda}_h(\gamma), \tilde{u}_h(\gamma)) = 0, \tag{8.3}$$

$$|\tilde{\lambda}_h(\gamma) - \lambda(\gamma)| + \|\tilde{u}_h(\gamma) - \widehat{\Pi}_h u(\gamma)\|_{H_0^1} \leqslant K_0(\lambda_0, u_0)\|u(\gamma) - \widehat{\Pi}_h u(\gamma)\|_{H_0^1}, \tag{8.4}$$

$$|\tilde{\lambda}_h(\gamma) - \lambda(\gamma)| + \|\tilde{u}_h(\gamma) - u(\gamma)\|_{H_0^1} \leqslant K_1(\lambda_0, u_0)\|u(\gamma) - \widehat{\Pi}_h u(\gamma)\|_{H_0^1}. \tag{8.5}$$

The constants $K_0(\lambda_0, u_0)$ and $K_1(\lambda_0, u_0)$ are independent of h and $\gamma \in [u_0(x_0) - \varepsilon_0, u_0(x_0) + \varepsilon_0]$.

Proof. The manner of the proof is exactly same to that of Theorem 7.3. We divide the proof into several steps.

Step 1: It follows from Lemma 3.2 and (8.2) that $DH(\gamma, \lambda, u) \in \mathscr{L}(\mathbb{R}^2 \times H_0^1, \mathbb{R} \times H^{-1})$ and

$$\begin{aligned}&\mathbb{R} \times \Lambda \times W_0^{1,\infty} \ni (\gamma, \lambda, u) \mapsto DH(\gamma, \lambda, u) \in \mathscr{L}(\mathbb{R}^2 \times H_0^1, \mathbb{R} \times H^{-1}) \text{ is}\\ &\qquad \text{Lipschitz continuous on bounded subset.}\end{aligned} \tag{8.6}$$

We, moreover, claim that, if $D_\lambda F(\lambda, u) \neq 0$ and $D_{(\lambda,u)}H(\gamma, \lambda, u) \in \mathscr{L}(\mathbb{R} \times W_0^{1,\infty}, \mathbb{R} \times W^{-1,\infty})$ is an isomorphism at $(\gamma, \lambda, u) \in \mathbb{R} \times \mathscr{R}(F, \mathscr{S})$, $D_{(\lambda,u)}H(\gamma, \lambda, u)$ is an isomorphism from $\mathbb{R} \times H_0^1$ to $\mathbb{R} \times H^{-1}$ as well.

Let $\tilde{H} := D_{(\lambda,u)}H(\gamma,\lambda,u) \in \mathscr{L}(\mathbb{R} \times H_0^1, \mathbb{R} \times H^{-1})$. Then

$$\tilde{H}(\mu,\psi) = (\psi(x_0), DF(\lambda,u)(\mu,\psi)), \quad \mu \in \mathbb{R},\ \psi \in H_0^1.$$

We have to consider two cases.

Suppose that we are in Case 1, that is, $D_uF(\lambda,u) \in \mathscr{L}(W_0^{1,\infty}, W^{-1,\infty})$ is an isomorphism. In this case, from the argument of Step 1 of Theorem 7.3, we know that $D_uF(\lambda,u) \in \mathscr{L}(H_0^1, H^{-1})$ is an isomorphism. Thus, we can prove our claim by the exactly same manner of the proof of Lemma 8.1.

Next, suppose that we are in Case 2, that is, $\dim \operatorname{Ker} Q = 1$ and $R \notin \operatorname{Im} Q$, where $Q := D_uF(\lambda,u) \in \mathscr{L}(W_0^{1,\infty}, W^{-1,\infty})$ and $R := D_\lambda F(\lambda,u)$. To avoid confusion we denote $D_uF(\lambda,u) \in \mathscr{L}(H_0^1, H^{-1})$ by $\tilde{Q}$.

We first show that $\operatorname{Ker} Q = \operatorname{Ker} \tilde{Q}$. Obviously, we have $\operatorname{Ker} Q \subset \operatorname{Ker} \tilde{Q}$. Let $\psi \in \operatorname{Ker} \tilde{Q} \subset H_0^1$. Then, we have

$${}_2\langle \tilde{Q}\psi, v\rangle_2 = \int_J [\alpha(x)\psi' v' + \beta(x)\psi v] \mathrm{d}x = 0, \quad \forall v \in H_0^1,$$

where $\alpha(x) := a_y(\lambda, x, u'(x))$ and $\beta(x) := f_y(\lambda, x, u(x))$. Hence, by a simple computation, we conclude that $\psi \in W^{2,p^*} \cap H_0^1 \subset W_0^{1,\infty}$. Hence, $\operatorname{Ker} Q = \operatorname{Ker} \tilde{Q}$. By Theorem 4.3, we know that $\operatorname{ind} \tilde{Q} = 0$. Thus, we obtain $\dim \operatorname{Coker} \tilde{Q} = 1$.

Next, we want to show that $D_\lambda F(\lambda,u) \notin \operatorname{Im} \tilde{Q}$. If $D_\lambda F(\lambda,u) \in \operatorname{Im} \tilde{Q}$, there exists some $\psi_1 \in H_0^1$ such that

$$\int_J \alpha(x)\psi_1' v' \,\mathrm{d}x = {}_2\langle D_\lambda F(\lambda,u), v\rangle_2 - \int_J \beta(x)\psi_1 v \,\mathrm{d}x, \quad \forall v \in H_0^1.$$

Again, by a simple computation, we conclude that ψ_1 is in the domain of Q. This is a contradiction because we assumed $D_\lambda F(\lambda,u) \notin \operatorname{Im} Q$.

Since we showed that $\dim \operatorname{Ker} \tilde{Q} = 1$ and $D_\lambda F(\lambda,u) \notin \operatorname{Im} \tilde{Q}$, we can prove our claim in the same way as in the proof of Lemma 8.1.

Step 2: We prepare several inequalities which we use later.

By the implicit function theorem, there exist $\varepsilon_1 > 0$ and a unique map

$$(u_0(x_0) - \varepsilon_1, u_0(x_0) + \varepsilon_1) \ni \gamma \mapsto (\lambda(\gamma), u(\gamma)) \in \Lambda \times W_0^{1,\infty}$$

such that $(\lambda_0, u_0) = (\lambda(\gamma_0), u(\gamma_0))$ with $\gamma_0 := u_0(x_0)$, $u(\gamma)(x_0) = \gamma$, and $F(\lambda(\gamma), u(\gamma)) = 0$.

As in Step 2 of the proof of Theorem 7.3, we easily obtain the following inequalities:

$$|\lambda(\gamma^*) - \lambda(\gamma)| + \|\hat{\Pi}_h u(\gamma^*) - \hat{\Pi}_h u(\gamma)\|_{H_0^1} \leqslant C_{15}|\gamma^* - \gamma|, \quad \forall \gamma^*, \quad \gamma \in I_1, \tag{8.7}$$

where $I_1 := [u_0(x_0) - \varepsilon_1/2, u_0(x_0) + \varepsilon_1/2]$.

Let $\alpha_0(x) := a_y(\lambda_0, x, u_0'(x))$. Again, we can assume, without loss of generality, that

$$a_y(\lambda_0, x, u_0'(x)) \geqslant \exists \delta_0 > 0.$$

Then, we define the canonical projection $\Pi_h^0 : H_0^1 \to \mathring{S}_h$, the isomorphism $T_0 \in \mathscr{L}(H^{-1}, H_0^1)$, and the C^2 map $\bar{F}_h^0 : \Lambda \times W_0^{1,\infty} \to W^{-1,\infty}$ as in Step 2 of the proof of Theorem 7.3. Of course, we have (7.11) and (7.12).

Now, define $\bar{H}_h^0 : \mathbb{R} \times \mathscr{S} \to \mathbb{R} \times W^{-1,\infty}$ by

$$\bar{H}_h^0(\gamma,\lambda,u) := (u(x_0) - \gamma, \bar{F}_h^0(\lambda,u)), \quad \gamma \in \mathbb{R}, (\lambda,u) \in \mathscr{S}. \tag{8.8}$$

As in Step 2 of the proof of Theorem 7.3, we observe

$$\begin{aligned}\|\overline{H}_h^0(\gamma,\lambda(\gamma),\widehat{\Pi}_h u(\gamma))\|_{\mathbb{R}\times H^{-1}} &= |\widehat{\Pi}_h u(\gamma)(x_0)-\gamma| + \|\overline{F}_h^0(\lambda(\gamma),\widehat{\Pi}_h u(\gamma))\|_{H^{-1}}\\ &\leqslant C_{16}h, \quad \forall\gamma\in I_1.\end{aligned}$$

Thus, we conclude that

$$\lim_{h\to 0}\sup_{\gamma\in I_1}\{h^{-1/2}\|\overline{H}_h^0(\gamma,\lambda(\gamma),\widehat{\Pi}_h u(\gamma))\|_{\mathbb{R}\times H^{-1}}\} = 0. \tag{8.9}$$

Step 3: We claim that there exist a positive $\varepsilon_2>0$ and a constant C_{17} independent of $h>0$ and $\gamma\in[u_0(x_0)-\varepsilon_2, u_0(x_0)+\varepsilon_2]$ such that

$$\|D_{(\lambda,u)}\overline{H}_h^0(\gamma,\lambda(\gamma),\widehat{\Pi}_h u(\gamma))(\mu,v_h)\|_{\mathbb{R}\times H^{-1}} \geqslant C_{17}(|\mu|+\|v_h\|_{H_0^1}), \quad \forall\mu\in\mathbb{R}, \quad \forall v_h\in\mathring{S}_h. \tag{8.10}$$

First, we remark that, by (8.6) and Step 1, the mapping

$$(u_0(x_0)-\varepsilon_1, u_0(x_0)+\varepsilon_1)\ni\gamma\mapsto(D_{(\lambda,u)}H(\gamma,\lambda(\gamma),u(\gamma)))^{-1}\in\mathscr{L}(\mathbb{R}\times H^{-1},\mathbb{R}\times H_0^1)$$

is continuous. Thus, we set

$$\omega := \max_{\gamma\in I_1}\|(D_{(\lambda,u)}H(\gamma,\lambda(\gamma),u(\gamma)))^{-1}\|_{\mathscr{L}(\mathbb{R}\times H^{-1},\mathbb{R}\times H_0^1)}.$$

Next, we write

$$\begin{aligned}D_{(\lambda,u)}\overline{H}_h^0(\gamma,\lambda(\gamma),\widehat{\Pi}_h u(\gamma))(\mu,v_h) &= (v_h(x_0), DF(\lambda(\gamma),u(\gamma))(\mu,v_h))\\ &\quad + (0,(D\overline{F}_h^0(\lambda(\gamma),\widehat{\Pi}_h u(\gamma)) - DF(\lambda(\gamma),u(\gamma)))(\mu,v_h)).\end{aligned} \tag{8.11}$$

On the first term of the right-hand side of (8.11), we have

$$\|(v_h(x_0), DF(\lambda(\gamma),u(\gamma))(\mu,v_h))\|_{\mathbb{R}\times H^{-1}} \geqslant \omega^{-1}(|\mu|+\|v_h\|_{H_0^1}). \tag{8.12}$$

On the second term of the right-hand side of (8.11), we write

$$\begin{aligned}&(D\overline{F}_h^0(\lambda(\gamma),\widehat{\Pi}_h u(\gamma)) - DF(\lambda(\gamma),u(\gamma)))(\mu,v_h)\\ &\quad= -\mu(I-P_h^0)D_\lambda F(\lambda(\gamma),u(\gamma))^{(a)}\\ &\qquad + \mu P_h^0(D_\lambda F(\lambda(\gamma),\widehat{\Pi}_h u(\gamma)) - D_\lambda F(\lambda(\gamma),u(\gamma)))^{(b)}\\ &\qquad + P_h^0(D_u F(\lambda(\gamma),\widehat{\Pi}_h u(\gamma)) - D_u F(\lambda(\gamma),u(\gamma)))v_h^{(c)}\\ &\qquad - (I-P_h^0)(-T_0^{-1}+D_u F(\lambda_0,u_0))v_h^{(d)}\\ &\qquad + (I-P_h^0)(D_u F(\lambda_0,u_0) - D_u F(\lambda(\gamma),u(\gamma)))v_h^{(e)}.\end{aligned} \tag{8.13}$$

Let us check each term of (8.13). For (a), we have

$$\|\mu(I-P_h^0)D_\lambda F(\lambda(\gamma),u(\gamma))\|_{H^{-1}} \leqslant \varepsilon(h)|\mu|, \tag{8.14a}$$

with $\lim_{h\to 0}\varepsilon(h)=0$ because $\|(I-P_h^0)D_\lambda F(\lambda(\gamma),u(\gamma))\|_{H^{-1}}\to 0$ as $h\to 0$ uniformly with respect to γ on $I_1=[u_0(x_0)-\frac{1}{2}\varepsilon_1, u_0(x_0)+\frac{1}{2}\varepsilon_1]$.

For (b), we easily obtain

$$\|\mu P_h^0(D_\lambda F(\lambda(\gamma),\widehat{\Pi}_h u(\gamma)) - D_\lambda F(\lambda(\gamma),u(\gamma)))\|_{H^{-1}} \leqslant C_{18}h^{1-1/p^*}|\mu|. \tag{8.14b}$$

For (c) and (d), we immediately get (see Step 3 of the proof of Theorem 7.3)

$$\|P_h^0(D_u F(\lambda(\gamma),\widehat{\Pi}_h u(\gamma) - D_u F(\lambda(\gamma),u(\gamma)))v_h\|_{H^{-1}} \leqslant C_{19}h^{1-1/p^*}\|v_h\|_{H_0^1}, \tag{8.14c}$$

$$\lim_{h\to 0}\|(I-P_h^0)(-T_0^{-1}+D_u F(\lambda_0,u_0))\|_{\mathscr{L}(H_0^1,H^{-1})} = 0. \tag{8.14d}$$

For (e), by (8.6), there exists a constant C_{20} such that

$$\|D_u F(\lambda(\gamma),u(\gamma)) - D_u F(\lambda(\gamma^*),u(\gamma^*))\|_{\mathscr{L}(H_0^1,H^{-1})} \leqslant C_{20}|\gamma-\gamma^*|$$

for any $\gamma,\gamma^*\in I_1$. Take $\varepsilon_2>0$ so that $\varepsilon_2^{-1} > 2\omega C_{20}\sup_{h>0}\|I-P_h^0\|_{\mathscr{L}(H^{-1},H^{-1})}$. Then, we have

$$\|(I-P_h)(D_u F(\lambda_0,u_0) - D_u F(\lambda(\gamma),u(\gamma)))\|_{\mathscr{L}(H^{-1},H^{-1})} \leqslant \tfrac{1}{2}\omega^{-1}, \tag{8.14e}$$

for any $\gamma\in[u_0(x_0)-\varepsilon_2,u_0(x_0)+\varepsilon_2]$.

From (8.12) and (8.14), we obtain

$$\|D_{(\lambda,u)}\overline{H}_h^0(\gamma,\lambda(\gamma),\widehat{\Pi}_h u(\gamma))(\mu,v_h)\|_{\mathbb{R}\times H^{-1}} \geqslant (\tfrac{1}{2}\omega^{-1}-\delta(h))(|\mu|+\|v_h\|_{H_0^1}),$$

with $\lim_{h\to 0}\delta(h)=0$. Therefore, we prove the claim (8.10) for sufficiently small $h>0$.

Step 4: Again, we prepare a few inequalities. By (8.8), we see

$$\|D_\gamma\overline{H}_h^0(\gamma,\lambda(\gamma),\widehat{\Pi}_h u(\gamma))\|_{\mathbb{R}\times H^{-1}} = \|(-1,0)\|_{\mathbb{R}\times H^{-1}} = 1. \tag{8.15}$$

Also, we immediately obtain

$$\begin{aligned}&\|D\overline{H}_h^0(\gamma^*,\lambda^*,u^*) - D\overline{H}_h^0(\gamma,\lambda(\gamma),\widehat{\Pi}_h u(\gamma))\|_{\mathscr{L}(\mathbb{R}^2\times H_0^1,\mathbb{R}\times H^{-1})}\\ &\quad\leqslant C_{21}(|\gamma^*-\gamma|+|\lambda^*-\lambda(\gamma)|+\|u^*-\widehat{\Pi}_h u(\gamma)\|_{W_0^{1,\infty}}),\end{aligned} \tag{8.16}$$

where $C_{21}=C_{21}(|\gamma^*|,|\lambda^*|,|\lambda|,\|u^*\|_{W_0^{1,\infty}})$.

Thus, by the inverse inequality (7.5) and (8.16), there exists a monotonically increasing function $L_2:\mathbb{R}^+\to\mathbb{R}^+$ independent of h such that, for all $\gamma,\gamma^*\in[u_0(x_0)-\frac{1}{2}\varepsilon_1,u_0(x_0)+\frac{1}{2}\varepsilon_1]$, $\lambda^*\in\Lambda$ and $v_h^*\in\mathring{S}_h$ with

$$h^{-1/2}(|\gamma^*-\gamma|+|\lambda^*-\lambda(\gamma)|+\|v_h^*-\widehat{\Pi}_h u(\gamma)\|_{H_0^1})\leqslant\xi,$$

we have

$$\begin{aligned}&\|D\overline{H}_h^0(\gamma^*,\lambda^*,u^*) - D\overline{H}_h^0(\gamma,\lambda(\gamma),\widehat{\Pi}_h u(\gamma))\|_{\mathscr{L}(\mathbb{R}^2\times H_0^1,\mathbb{R}\times H^{-1})}\\ &\quad\leqslant L_2(\xi)h^{-1/2}(|\gamma^*-\gamma|+|\lambda^*-\lambda(\gamma)|+\|v_h^*-\widehat{\Pi}_h u(\gamma)\|_{H_0^1}).\end{aligned} \tag{8.17}$$

Step 5: By (8.7), (8.9), (8.10), (8.15) and (8.17), we can apply Theorem 7.2 to the operator $\overline{H}_h^0$ in the following situation;

$X=\mathbb{R}$ with norm $h^{-1/2}|\gamma|$,

$Y=\mathbb{R}\times\mathring{S}_h\subset\mathbb{R}\times H_0^1$ with norm $h^{-1/2}(|\lambda|+\|v_h\|_{H_0^1})$,

$Z = \mathbb{R} \times \mathring{S}_h \subset \mathbb{R} \times H^{-1}$ with norm $h^{-1/2}(|\lambda| + \|T_0^{-1} v_h\|_{H^{-1}})$,
$S = [u_0(x_0) - \varepsilon_0, u_0(x_0) + \varepsilon_0]$ with $\varepsilon_0 := \min(\frac{1}{2}\varepsilon_1, \varepsilon_2)$,
$y(\gamma) = (\lambda(\gamma), \widehat{\Pi}_h u(\gamma))$.

Since $\|A_h\|_{\mathscr{L}(X \times Y, Z)} = \|A_h\|_{\mathscr{L}(\mathbb{R}^2 \times H_0^1, \mathbb{R} \times H^{-1})}$ for all $A_h \in \mathscr{L}(\mathbb{R}^2 \times \mathring{S}_h, \mathbb{R} \times \mathring{S}_h)$ and (8.9), there exists a unique C^2 function $[u_0(x_0) - \varepsilon_0, u_0(x_0) + \varepsilon_0] \ni \gamma \mapsto (\tilde{\lambda}_h(\gamma), \tilde{u}_h(\gamma)) \in \mathbb{R} \times \mathring{S}_h$ such that

$$\overline{H}_h^0(\gamma, \tilde{\lambda}_h(\gamma), \tilde{u}_h(\gamma)) = (0, 0), \tag{8.18}$$

and the inequality

$$|\tilde{\lambda}_h(\gamma) - \lambda(\gamma)| + \|\tilde{u}_h(\gamma) - \widehat{\Pi}_h u(\gamma)\|_{H_0^1} \leqslant C_{22} \|\overline{H}_h^0(\gamma, \lambda(\gamma), \widehat{\Pi}_h u(\gamma))\|_{\mathbb{R} \times H^{-1}} \tag{8.19}$$

holds for all $\gamma \in [u_0(x_0) - \varepsilon_0, u_0(x_0) + \varepsilon_0]$. It is clear that (8.18) implies (8.3). To get (8.4) and (8.5), we observe that

$$\begin{aligned} \|\overline{H}_h^0(\gamma, \lambda(\gamma), \widehat{\Pi}_h u(\gamma))\|_{\mathbb{R} \times H^{-1}} &= \|P_h^0(F(\lambda(\gamma), u(\gamma)) - F(\lambda(\gamma), \widehat{\Pi}_h u(\gamma)))\|_{H^{-1}} \\ &\leqslant C_{23} \|u(\gamma) - \widehat{\Pi}_h u(\gamma)\|_{H_0^1}. \end{aligned} \tag{8.20}$$

Therefore, combining (8.19) and (8.20), we obtain (8.4), (8.5), and complete the proof. □

By the same way as in Section 7, we obtain the following propositions.

Corollary 8.5. *Suppose that the assumptions of Theorem* 8.4 *hold. Then, there exists a constant* $K_2(\lambda_0, u_0) > 0$ *independent of* $h > 0$ *and* $\gamma \in [u_0(x_0) - \varepsilon_0, u_0(x_0) + \varepsilon_0]$ *such that*

$$|\lambda(\gamma) - \tilde{\lambda}_h(\gamma)| + \|u(\gamma) - \tilde{u}_h(\gamma)\|_{W_0^{1,\infty}} \leqslant K_2(\lambda_0, u_0) h^{1/2}.$$

Theorem 8.6. *Suppose that Assumption* 7.1 *holds for* $d \geqslant 2$. *Let* $\widetilde{\mathscr{M}}_0 \subset \mathscr{M}_0$ *be a connected compact subset with the following properties*:

(1) $D_\lambda F(\lambda, u) \neq 0$ *for any* $(\lambda, u) \in \widetilde{\mathscr{M}}_0$.

(2) *There exist* $x_0 \in J$ *such that* $D_{(\lambda,u)} H(\gamma, \lambda, u)$ *defined by* (8.2) *is an isomorphism for all* $(\lambda, u) \in \widetilde{\mathscr{M}}_0$.

Then $\widetilde{\mathscr{M}}_0$ *is parametrized by* $\gamma = u(x_0)$. *We assume, without loss of generality, that the above* x_0 *is a nodal point of* $\mathring{S}_h$ *for all sufficiently small* $h > 0$.

Then there exists the corresponding finite element solution branch $\widetilde{\mathscr{M}}_h \subset \mathscr{M}_h$ *which is parametrized by the same* γ, *that is,* $u_h(\gamma)(x_0) = \gamma$ *and* $F_h(\lambda_h(\gamma), u_h(\gamma)) = 0$ *for any* γ.

Moreover, we have

$$|\lambda(\gamma) - \lambda_h(\gamma)| + \|\widehat{\Pi}_h u(\gamma) - u_h(\gamma)\|_{H_0^1} \leqslant K_3 \|u(\gamma) - \widehat{\Pi}_h u(\gamma)\|_{H_0^1},$$

$$|\lambda(\gamma) - \lambda_h(\gamma)| + \|u(\gamma) - u_h(\gamma)\|_{H_0^1} \leqslant K_4 \|u(\gamma) - \widehat{\Pi}_h u(\gamma)\|_{H_0^1},$$

$$|\lambda(\gamma) - \lambda_h(\gamma)| + \|u(\gamma) - u_h(\gamma)\|_{W_0^{1,\infty}} \leqslant K_5 h^{1/2},$$

$$\widetilde{\mathscr{M}}_h \subset \mathscr{R}(F, \mathscr{S}),$$

for all $\gamma = u(x_0)$, $(\lambda(\gamma), u(\gamma)) \in \widetilde{\mathscr{M}}_0$, $(\lambda_h(\gamma), u_h(\gamma)) \in \widetilde{\mathscr{M}}_h$. *Here,* K_3, K_4, K_5 *are positive constants independent of* h *and* γ.

For the W_0^{1,p^*}-norm estimate, we have the following theorem as in Section 7.

Theorem 8.7. *Suppose that the assumptions of Theorem* 8.6 *and* (7.27) *hold. Then, for the corresponding finite element solution branch we have the following estimates*:

$$|\lambda(\gamma) - \lambda_h(\gamma)| + \|\widehat{\Pi}_h u(\gamma) - u_h(\gamma)\|_{W_0^{1,p^*}} \leqslant K_6 \|u(\gamma) - \widehat{\Pi}_h u(\gamma)\|_{W_0^{1,p^*}},$$

$$|\lambda(\gamma) - \lambda_h(\gamma)| + \|u(\gamma) - u_h(\gamma)\|_{W_0^{1,p^*}} \leqslant K_7 \|u(\gamma) - \widehat{\Pi}_h u(\gamma)\|_{W_0^{1,p^*}},$$

$$|\lambda(\gamma) - \lambda_h(\gamma)| + \|u(\gamma) - u_h(\gamma)\|_{W_0^{1,\infty}} \leqslant K_8 h^{1-(1/p^*)},$$

for all $\gamma = u(x_0)$, $(\lambda(\gamma), u(\gamma)) \in \widetilde{\mathscr{M}}_0$, $(\lambda_h(\gamma), u_h(\gamma)) \in \widetilde{\mathscr{M}}_h$. *Here,* K_6, K_7, K_8 *are positive constants independent of* h *and* γ.

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